Attraction and Adhesion of a Charged Insulative Toner Particle

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Abstract

The force attracting a charged, insulative particle to a nearby surface is often calculated by assuming that the total charge is concentrated at the center of the particle. While this is true when the particle is far away, there is evidence that a different model should be used when the particle is very close, or in contact. This attractive force is calculated in terms of the bispherical coordinate system, which provides an analytical solution valid when the particle approaches and contacts the surface. Comparison of this model with other experimental and analytical work is carried out to determine the validity of the various models, and to provide approximate formulas useful in engineering work. This work indicates that electrostatic adhesive force may be larger than normally expected when the surface charging is larger than traditional assumption, as reported in several experiments.

Introduction

The toner used in electrophotography carries a charge that leads to attraction toward other objects, and eventually to adhesion when it comes in contact. The electrostatic force is a relatively long-range effect that dominates when the particle is several diameters away, but it is just one of many possible forces of adhesion once contact is made. These other forces, including van der Waals force and surface tension, have been described in several reviews [1, 2].

Electrostatic adhesion (and subsequent detachment) of toner is the basis for all electrophotographic development engines, so it is obviously important to identify the dominant effects so as to understand how to control and modify the development process. One approach is to compare experimental results to the predictions of a theoretical model for the force.

The most common model of the electrostatic force on the toner replaces the particle with a point charge at its center, leading to an adhesion force that is independent of the particle's electrical properties. This adhesion force, however, is often smaller than experimental adhesion values, often by orders of magnitude [3]. Several explanations for the failure of the central point charge model have been offered.

If the particle has a finite dielectric constant, polarization effects [4], will increase the force exerted by the charge. Another possibility is a patchy charge distribution related to surface roughness [5], which tends to concentrate the charge at the point of contact. Or, it may be that the van der Waals forces are so large that they can overwhelm the electrostatic force [6].

Before deciding on which mechanisms play a role in toner adhesion, it would be well to have an analytical model of the electrostatic force that is both well-defined and useful for toner charging. Previous models often assume that the charge is located at the center of the toner particle, whereas charging by triboelectrification implies that the charge will be on the surface. Some neglect the effect of the particle's permittivity, which distorts the field near the contact. Others have described complicated infinite summations that are difficult to apply quickly and correctly in an engineering context.

The present paper presents an alternative approach to calculating the electrostatic adhesion force. It is based on a rather complicated analytical solution for the electric fields around a charged particle near a wall, but it presents the adhesion force as a short analytical expression that gives the force for any size or dielectric constant of the particle.

Force Calculation

The electrostatic force on the particle is obtained by integration of the Maxwell stress tensor, which requires knowledge of the electric fields around the particle. These are obtained by solution of Laplace's equation in bispherical coordinates. This technique has been used in the past to calculate forces on insulative [7] and conductive [8] particles near a wall, but has not been applied to particles with a surface charge distribution, which is the situation for toner.

The toner particle is assumed to be an insulative sphere of radius R and dielectric constant κ_s, with a total charge Q that is distributed uniformly across its surface, as shown in Figure 1. The sphere is immersed in an ambient medium (normally air)

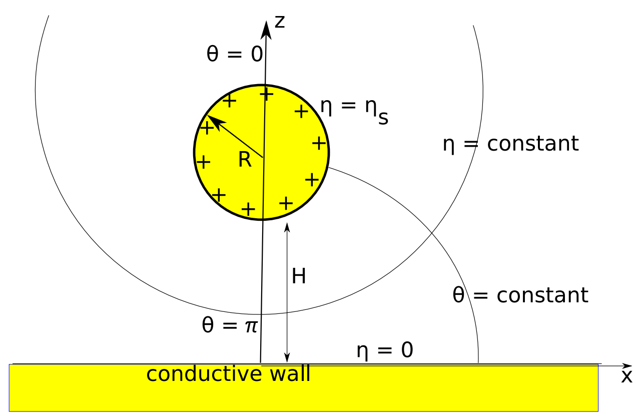

Figure 1. *The bipolar coordinate system (bipolar.svg)*

with a dielectric constant of κ_a, and separated from the conducting ground plane by a distance H. In this paper, dimensioned quantities will be written as upper case letters, and the corresponding dimensionless quantities in lower case. Thus the normalized spacing will be given by $h = H/R$.

Figure 1 also shows surfaces in the bispherical coordinate system that will be used to calculate the force. The surface of the sphere is at $\eta = \eta_s$, and the ground plane is at $\eta = 0$. The θ coordinate describes distances along the surfaces of the sphere and plane.

The electrostatic field is obtained by solving Laplace's equation subject to the boundary conditions imposed at the ground plane and the sphere. The solution technique, which is outlined in the appendix, leads to an analytical expression for the force, in the form of a convergent infinite series.

Attraction toward the wall

In bispherical coordinates, the particle is separated from the wall by the distance h, so the basic result is the attractive force, which is described in this section. In the following section, the adhesive force will be obtained by taking the limit of very small spacing. The attractive force is also useful in its own right, since it influences the trajectories of particles moving near the wall, and it may represent the apparent adhesive force if the particle is prevented from actually touching the wall.

Comparison with earlier attraction models

Figure 2 show the variation of the attractive force as a function of spacing for particles with dielectric constants ranging from $\kappa = 0$ to $\kappa = \infty$. Two special cases, $\kappa = 1$ and $\kappa = \infty$ are highlighted by the presence of circular dots. These two cases have been described previously, so they serve as a check on the present model.

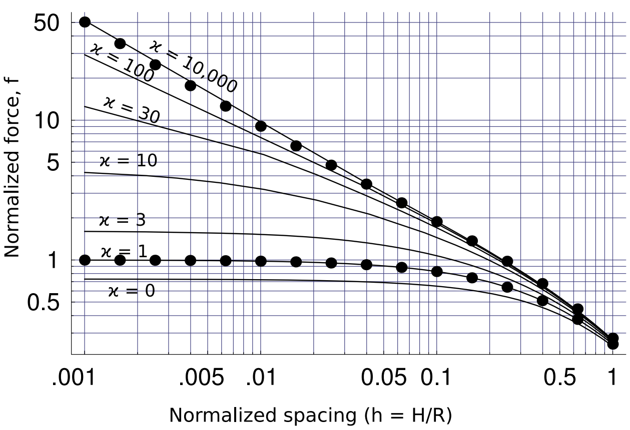

Figure 2. *Electrostatic attractive force as a function of spacing for various dielectric constants (nipattraction.svg)*

Perhaps the most common prior model is an image force calculation that assumes that the total charge is concentrated at the center of the sphere. This model neglects the effect of the dielectric constant of the particle. In the present model, it is represented by setting the relative dielectric constant of the particle to $\kappa = 1$, corresponding to the lower curve with circles in Figure 2. The image force for the central charge model is

$$F_{\mathrm{i}} = \frac{Q^2}{16\pi\kappa_{\mathrm{a}}\varepsilon_0 R^2}\, f_{\mathrm{i}}(h) \tag{1}$$

where

$$f_{\mathrm{i}}(h) = \frac{1}{(1+h)^2} \tag{2}$$

is the normalized attractive force, and $h = H/R$ is the normalized spacing. This force is shown by the points in the figure, and it is clear that the two models are in complete agreement. In particular, both models predict that the adhesive force (at the point of contact) is finite and equal to

$$F_{\mathrm{i}} = \frac{Q^2}{16\pi\kappa_{\mathrm{a}}\varepsilon_0 R^2} \tag{3}$$

at least when the dielectric constant is unity. This result is identical to the central-charge approximation.

The upper dotted curve in Figure2 shows the attractive force for an infinite dielectric constant (actually $\kappa = 10^4$). It is well known that the electrostatic fields outside an object with infinite permittivity are identical to those outside a perfect conductor, so this case can be compared to previous work [8] that found a good approximation to the force on a charged conducting sphere as

$$f_{\mathrm{c}}(h) \approx \frac{1}{\left(h+h^2\right)\,\left(1+\frac{1}{2}\log\left(1+1/h\right)\right)^2} \tag{4}$$

using the same normalization for the force. This force on a conducting sphere is shown by circular dots along the upper curve, which again are in agreement with the present model. Thus the results of the bispherical solution agree with the earlier limiting cases, giving us some confidence in their validity.

Notice that the force on a charged conductor (or infinite dielectric) continues to rise as the sphere gets closer to the ground plane, and reaches infinity at the point of contact. In this case, it is not possible to define an unambiguous electrostatic adhesive force.

When the particle is far from the wall, the attraction force is independent of the dielectric constant, and all the force calculation merge in the lower right of Figure 2, corresponding to an attractive force of

$$F_{\mathrm{i}} = \frac{Q^2}{16\pi\kappa_{\mathrm{a}}\varepsilon_0 R^2}\,\frac{1}{h^2} = \frac{Q^2}{16\pi\kappa_{\mathrm{a}}\varepsilon_0 H^2} \tag{5}$$

This region is of little interest for adhesion, but plays an important role in calculations of toner trajectory.

Effect of toner dielectric constant

Practical insulative toners have dielectric constants greater than unity and less than infinity, so the two special cases discussed above are not directly applicable for toner force calculations. For that, we can consider normalized force curves for some intermediate dielectric constants ($\kappa = 3, 10, 30, 100$) that are also presented in Figure 2. The curves show that the force on the particle will always be greater than that implied by the central charge model. This effect is greater as the dielectric constant increases, and as the particle comes closer to the wall. Physically, the force increases because more of the field is channeled through the particle to go directly to the ground plane, rather than through the ambient medium. This leads to a more intense electric field in the gap, and thus to a stronger force. Similar curves were presented by Fowlkes and Robinson [4] some time ago, for a 1-cm sphere. The force was also calculated by Davis [7] in connection with water droplets with a charge at the center. The present paper extends those results by normalizing the results so that they can more easily be applied to particles with arbitrary size and spacing.

Nakajima [9] also found a similar result, based on a model of two spheres, one much larger than the other. His results also appear to be close to the present solution, which uses a flat plane rather than a large sphere.

The lowest curve in Figure 2 shows the force when the particle has a dielectric constant that is much less than the ambient medium. This could occur if the particle were a hollow shell in a liquid or solid medium. The force is less than that for the other cases, but is still finite. It should be noted however, that this is the normalized force, and an increase in the dielectric constant of the ambient medium will lead to a smaller attractive force, in real terms. In any event, this case holds little interest for current toners.

Adhesion to the wall

Using the analytical expression for the attraction force, we can now proceed to find the adhesive force as the limit of the attractive force when the particle touches the ground plane, or can not get any closer to it.

Low dielectric constant

Most insulative toner has a relatively low dielectric constant. The electrostatic adhesive force for dielectric constants below 10 is shown in Figure 3 as a function of the particle's dielectric constant. The small filled circles in this figure represent the adhesive force calculated with the bipolar model, as described in the Appendix. The four larger dots represent values obtained from a finite-element approach by Feng and Hays [10] for a similar problem having additional walls to bound the solution region. The agreement between the numerical and analytical results is encouraging.

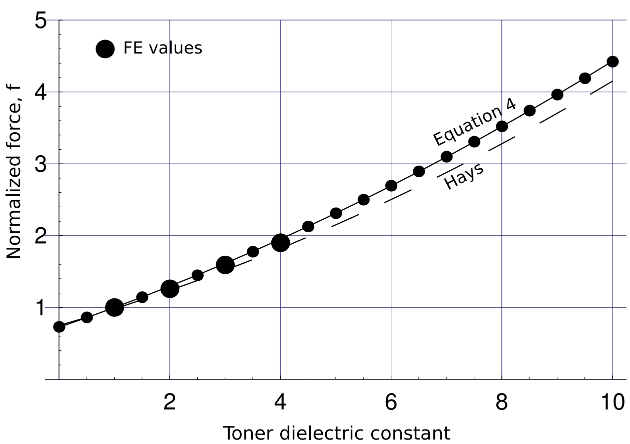

Figure 3. *Adhesive force as a function of dielectric constant (lokappaallP.svg)*

For common values of toner dielectric constant (from 2 to 10), the adhesive force increases gradually with the dielectric constant. The variation in force can be fitted fairly well by a function of the form

$$f_{\kappa\mathrm{low}}\left(\kappa\right) \approx 0.20\,\kappa + 0.70\exp\,\left(0.12/\kappa\right) \tag{6}$$

which is valid for the range $0 \le \kappa \le 10$. The fitted function is shown by the solid line in Figure 3. Since the agreement appears to be quite good, this equation will be used to calculate the electrostatic adhesion force for a typical toner particle in a following section.

Hays and Sheflin [11] have given an approximation to the adhesion force that includes the effect of surface coverage of the toner, based on multipole expansions. Their result (in the limit of zero coverage) is

$$f_{\mathrm{FE}}(\kappa) \approx 0.75 + 0.22\kappa + 0.012\kappa^2 \tag{7}$$

It is also shown in Figure 3 as a dashed line. It is not as close to the analytical solution as Equation 6, but of course it includes more variables than the present model, and is based on a slightly different geometry.

High dielectric constant

Many particles of general interest, will have higher dielectric constants. The normalized electrostatic adhesive force, as calculated from the bispherical model, is shown by the smaller dots in Figure 4 as a function of the dielectric constant. As expected from the previous discussion, the adhesive force continues to increase gradually with the dielectric constant. The force grows slowly at first, and then more rapidly to a value significantly higher than the image charge approximation.

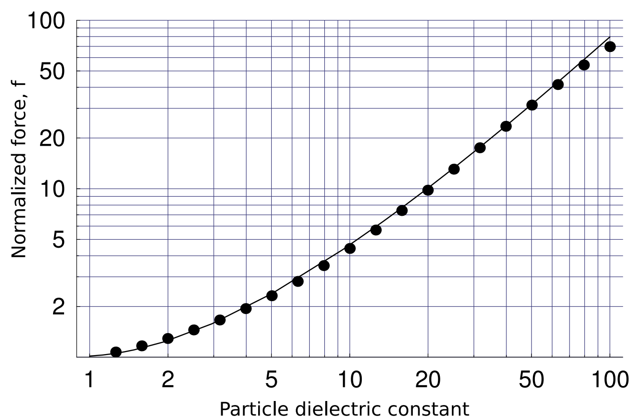

Figure 4. *Electrostatic adhesive force for large dielectric constants (fitforceallkappasimpF.svg)*

Although the variation is well behaved, the actual calculations are complex and lengthy. As before, we approximate them by a function that is much more useful in practical work. This function, which is shown as the solid line in Figure 4, is given by

$$f_{\kappa\mathrm{high}}\left(\kappa\right) \approx 0.06\,\kappa^{3/2}\,\exp\left(\sqrt{8/\kappa}\right) \tag{8}$$

It gives very good agreement over the entire range of dielectric constants ($1 \le \kappa \le 100$), and is relatively simple, compared to the calculations described in the Appendix. The presence of integers in the functional dependence is interesting, since it suggests some underlying simplification may be possible.

Effect of spacers

In some toner systems, the particles are prevented from making physical contact with the substrate, perhaps by coating them with an insulating additive powder like silica [12]. In that case, the electrostatic "adhesive" force will be smaller, and will not reach infinity even for very large dielectric constants. This behavior is illustrated in Figure 5. for final spacings ranging from $h = 0.1$ to

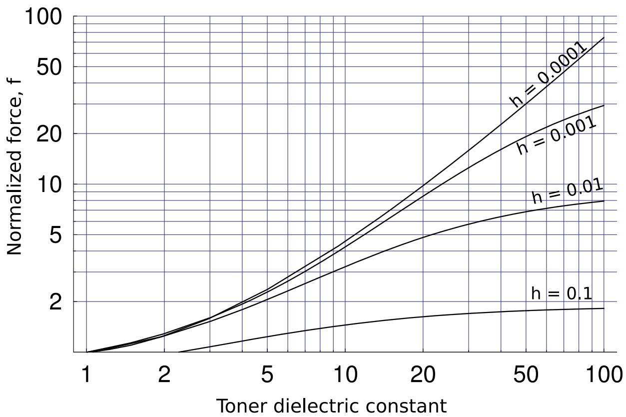

Figure 5. *Adhesive force as a function of dielectric constant when the spacing remains finite (nipvskappaall.svg)*

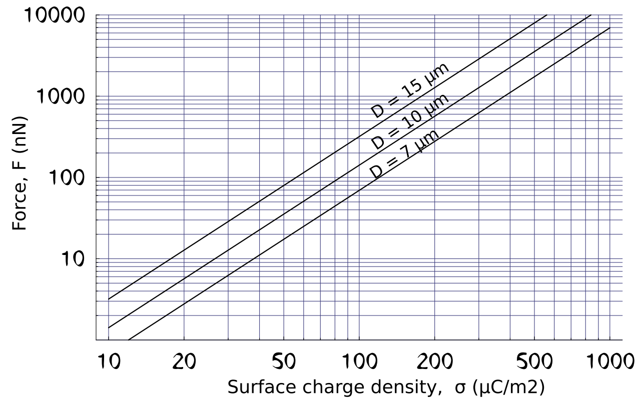

Figure 6. *Adhesive force as a function of surface charge density for various toner diameters (nipphysicalradius.svg)*

$h = 0.0001$. The closest spacing (0.0001) would represent a physical spacing of 0.5 nm for a particle with a diameter of 10 µm.

If the spacing is relatively large ($h = 0.1$), the force will not be much bigger than the image force, even for very large dielectric constants. For closer spacing, the force is much bigger, but eventually reaches a limiting value that depends on the dielectric constant. Most of the plastics that are used in insulative toner have dielectric constants below 4, so the force reduction will only be substantial if the spacing is greater than about $h \approx 0.01$. In other words, a spacing of 50 nm will not appreciably decrease the electrostatic adhesion force on a 10 µm particle.

Examples of toner adhesion

As an example of the approximations described above, consider the adhesion of a typical toner particle, with a dielectric constant of $\kappa = 3$, surrounded by air. Since the particle becomes charged by triboelectric effects, the charge tends to be proportional to the surface area, so it is a more physical approach to describe the force in terms of average surface charge density, which is related to the total charge by

$$Q = 4\pi R^2 \sigma \tag{9}$$

Using this relation in the force expression (Equation 21 of the Appendix) gives

$$F = \frac{\pi R^2 \sigma^2}{\kappa_a \varepsilon_0} f(\kappa) \tag{10}$$

With a given dielectric constant, the electrostatic adhesive force will depend on the radius and the surface charge density. A plot of the force as a function of charge density is given in Figure 6 for toners with diameters of 7, 10, and 15 μm. The force increases with particle size, for a given charge density. The charge densities shown in the figure range from 10 to 1000 $\mu\text{C/m}^2$, with the corresponding forces reaching to well over 1000 nN. At first glance, this might appear too high for practical consideration, since the breakdown strength of air is often quoted as 3 V/m, equivalent to a surface charge density of about 25 $\mu\text{C/m}^2$. In the micron size range, however, much larger charge levels are possible, and even common. For gaps smaller than 5 µm, such as those in development and transfer nips, the breakdown field is on the order of 68 V/m (600 $\mu\text{C/m}^2$) [13].

Larger charge densities are not confined to near-contact situations. Air breakdown requires a large electrical field over a certain distance to allow avalanches to grow, and strong fields exist only over a very short range near a small charged particle. Measurements of toner have shown relatively large charge densities, and the charge densities on smooth (liquid) particles with a diameter of 5 µm have been measured [14] at about 500 $\mu\text{C/m}^2$. Thus there is considerable experimental evidence that the charge on toner particles can be higher than 25 $\mu\text{m/m}^2$, and may be over an order of magnitude higher. Since the force is proportional to the square of the charge, the electrostatic force could, in principle, account for observed adhesion forces that are well over 100 nN.

Discussion

When the toner particle is far from the ground plane, the central point charge model is always valid, and would be appropriate for calculations of particle trajectories. Close to the wall, however, that model can only be valid when there is no difference between the ambient and particle dielectric constants, a situation that does not arise in electrophotographic development. For calculation of adhesion forces, the dielectric constant of the particle must always be taken into account.

The dielectric constant of the particle always leads to an increase in the adhesive force. For typical toner particles, this increase is a factor of 1.5 to 2 higher than that of the central-charge model, and would be even higher for larger dielectric constants. For any finite dielectric constant, however, the electrostatic adhesion force is finite, even though some of the charge on the surface is in "contact" with the grounded wall. The force does not become infinite there because uniform surface charge leads to a finite electrical field in the gap [5]. A different result might be expected if the charge distribution were modeled as a collection of point charges,[15], since a point charge, unlike a patch, will have an infinite force at contact. This points out one problem of numerical modeling near a singular point, and highlights the utility of analytical solutions for adhesive forces.

The model presented here describes a particle with a uniform surface charge. It does not account for patchy charge distributions due to surface roughness or trapped charges, which can be expected to increase the adhesion. It also assumes that the particle is insulative. In particular, it neglects the possibility that the particle surface is contaminated by moisture, which will increase the conductivity, and can also lead to increased adhesive force [16].

Conductivity effects occur over a characteristic time, so their importance will depend on the time that the particle will be close to the substrate. In a development system, this might be a fraction of a second. In a laboratory measurement of adhesive force, however, the time is likely to be much longer. If these time-dependent effects play a role, such measurements of adhesive forces may not be directly relevant to development design.

If the toner is blocked from reaching the substrate, the apparent adhesion force will be less than the force for true contact between the toner and substrate. The reduction is stronger when the particle has a higher dielectric constant. For typical toners, a substantial reduction in adhesion can be obtained by separations greater than $H \approx 0.01R$.

Finally, the approximate adhesion formulas derived here provide a much easier way to incorporate realistic adhesion calculations into toner and development design. These formulas are

$$F_\mathrm{i} = \frac{Q^2}{16\pi\kappa_\mathrm{a}\varepsilon_0 R^2}\begin{cases} 0.20\,\kappa + 0.70\exp\left(^{0.12}/_{\kappa}\right) & ,\ 0 \le \kappa \le 10 \\ 0.06\,\kappa^{3/2}\exp\left(\sqrt{^{8}/_{\kappa}}\right) & ,\ 1 \le \kappa \le 100 \end{cases} \tag{11}$$

Acknowledgments

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Appendix: Outline of force calculation

The solution for the force in the bipolar coordinate system is rather lengthy, so only an outline of the procedure can be given here. The sphere has a a radius R, a dielectric constant κ_s and carries a uniform surface charge density σ. The potential will be normalized to the potential on the isolated charged sphere, so that

$$\Phi = V_\mathrm{s}\phi = \frac{\sigma R}{\kappa_\mathrm{a}\varepsilon_0}\phi \tag{12}$$

where V_s is the potential on a charged sphere far from ground. The lowercase ϕ is the dimensionless potential. It must be zero at the ground plane, continuous across the surface of the sphere, and finite inside the sphere. The solution of Laplace's equation that satisfies these conditions takes the form [17]

$$\phi_\text{a} = (\cosh\eta - \cos\theta)^{1/2} \sum_{n=0}^{\infty} A_n \frac{\sinh[(n+1/2)\eta]}{\sinh[(n+1/2)\eta_\text{s}]} \text{P}_n(\cos\theta)$$
$$\phi_\text{p} = (\cosh\eta - \cos\theta)^{1/2} \sum_{n=0}^{\infty} A_n \frac{\exp[-(n+1/2)\eta]}{\exp[-(n+1/2)\eta_\text{s}]} \text{P}_n(\cos\theta) \tag{13}$$

where the subscripts (a,p) denote the solution in the ambient fluid and the particle.

In order to satisfy the last boundary condition, and to evaluate the force, the η- component of the electric field is needed. It is given by

$$E = -\frac{\cosh\eta - \cos\theta}{R\sinh\eta_\text{s}} \frac{\partial\phi}{\partial\eta} \tag{14}$$

At the surface of the sphere, there is a discontinuity in the normal electric field given by

$$\sum_{n=0}^{\infty} \left[\kappa E_{\text{p},n}(\eta_\text{s},\ \theta) - E_{\text{a},n}(\eta_\text{s},\ \theta)\right] = \frac{\sigma}{\kappa_\text{a}\varepsilon_0} \tag{15}$$

where σ is the uniform surface charge density on the sphere, given by

$$\sigma = \frac{Q}{4\pi R^2} \tag{16}$$

and $\kappa = \kappa_\text{p}/\kappa_\text{a}$ is the ratio of dielectric constants for the particle and the ambient.

The E- field is normalized as

$$E = \frac{\sigma}{\kappa_\text{a}\varepsilon_0} e \tag{17}$$

(The lower case e is not an exponential, but a normalized field.) This allows the boundary condition to be written as

$$\sum_{n=0}^{\infty} [A_n\ (a_n + b_n x)\ \text{P}_n(x)] = -\frac{2\sinh\eta_\text{s}}{\sqrt{\cosh\varepsilon_0 - x}} \tag{18}$$

where $x = \cos\theta$, and the coefficients a_n and b_n depend on η_s and κ, and are given by

$$a_n = \tfrac{1}{2} - (1+2n)\cosh(\eta_0)\ (\kappa + \coth[(n+1/2)\,\eta_\text{s}]) + (\kappa - 1)\sinh(\eta_0) \tag{19}$$
$$b_n = (1+2n)\ (\kappa + \coth[(n+1/2)\,\eta_\text{s}])$$

The coefficients A_n are obtained by the orthogonal expansion of the final boundary condition (Equation 15) in Legendre polynomials over the surface of the sphere ($-1 \leq x \leq 1$), giving an infinite set of equations of the form

$$\begin{bmatrix} a_0 & \frac{b_1}{3} & 0 & 0 & \cdots \\ \frac{b_0}{3} & \frac{a_1}{3} & \frac{2\,b_2}{15} & 0 & \cdots \\ 0 & \frac{2\,b_1}{15} & \frac{a_2}{5} & \frac{3\,b_3}{35} & \cdots \\ 0 & 0 & \frac{3\,b_2}{35} & \frac{a_3}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} = -2\sqrt{2}\sinh\eta_\text{s} \begin{bmatrix} e^{-\frac{\eta_\text{s}}{2}} \\ \frac{1}{3}e^{-\frac{3\eta_\text{s}}{2}} \\ \frac{1}{5}e^{-\frac{5\eta_\text{s}}{2}} \\ \frac{1}{7}e^{-\frac{7\eta_\text{s}}{2}} \\ \vdots \end{bmatrix} \tag{20}$$

In practice, the solution converges for a finite subset of the equations, as determined by examining the change in the result after increasing the number of equations by 50%. In this paper, a change of less than 0.01% was considered sufficient.

The force on the sphere is given by integrating the electrostatic stress tensor over a surface that encloses the sphere [18],

$$F = \frac{Q^2}{16\pi\kappa_\text{a}\varepsilon_0 R^2} f = \oint \frac{\varepsilon E^2}{2} \frac{R^2\sinh^2\eta_\text{s}\ \sin\theta}{(\cosh\eta - \cos\theta)^2} \text{d}\theta\text{d}\psi \tag{21}$$

where f is the normalized force. Here the surface was bounded by the ground plane $\eta = 0$, and extended to infinity, so the normalized force is given by

$$f = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} e_\text{g}^2(\theta) \frac{\sinh^2\eta_\text{s}\ \sin\theta}{(1-\cos\theta)^2} \text{d}\theta\text{d}\psi \tag{22}$$

The electric field at the ground plane is given by Equation 14 as

$$e_\text{g}(\eta,\theta) = -\frac{\text{csch}(\eta_\text{s})}{2}(1-\cos\theta)^{\frac{3}{2}} \sum_{n=0}^{\infty} G_n(\eta_\text{s},\kappa)\ \text{P}_n(\cos\theta) \tag{23}$$

where

$$G_n(\eta_\text{s},\kappa) = A_n(\eta_\text{s},\kappa)\ (1+2n)\ \text{csch}(\eta_\text{s}\ (n+1/2)) \tag{24}$$

is the coefficient in the ground-plane expansion for the field. Using the geometrical relations of the bispherical coordinate system allows us to replace η_s with

$$\eta_\text{s} = \text{acosh}(1+h) \tag{25}$$

Squaring the field, and carrying out the integral in Equation 22 gives the force as

$$f = \frac{1}{2}G_0^2 + \frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{2}{2n+1}G_n^2 - \frac{4n}{(2n+1)(2n-1)}G_{n-1}G_n\right] \tag{26}$$

This sum can be truncated with the same method and accuracy goal as the calculation of the A_n.