Error Diffusion: Recent Developments in Theory and Applications

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Abstract

Error diffusion is a popular technique for digital halftoning. The purpose of this paper is to illustrate the versatility of error diffusion with applications beyond halftoning and to show its connections to interesting mathematical questions. We present some recent developments in the theory and application of error diffusion and related algorithms. In particular, we illustrate the stability of error diffusion and present applications in a variety of areas besides image halftoning including scheduling, image processing, image watermarking and displays.

Introduction

Printers, both digital and analog, utilize a few colors to produce the wide variety of colors we see in printed images by means of halftoning. Halftoning is the art of producing an image with a few colors so that at a suitable distance the image appears to consist of many colors. In digital printers, the image is decomposed into a regular grid of pixels and digital halftoning algorithms produce a halftone image whose pixels comprise a few output colors such that it resembles an input image whose pixels comprise of many more colors from the input color space.

We describe the halftoning problem as follows. Given an *input* image I, where each pixel $I(i,j)$ at location (i,j) is a vector in the input color gamut S, the goal is to generate an output image O whose pixels $O(i,j)$ are vectors from a restricted set V of output colors such that the output image O resembles the input image I when viewed at some distance. In general the input color gamut S and the output colors V are chosen to be set of vectors in a color space such as *Lab*, *RGB* or *XYZ*.

Error diffusion, invented in 1975,[1] is a relatively fast technique for digital halftoning of images and generates high quality halftones. The basic idea of error diffusion is the following: errors occurred due to the discrepancy between the output pixel value and the input pixel value is propagated to neighboring pixels. At each pixel, the output pixel color is chosen based on the modified input pixel value, which is defined as the input pixel value plus the errors that were propagated from other pixels. More specifically[2]:

Definition 1 *An* error diffusion algorithm *is defined by the following steps*:

1. *Choose an enumeration of the pixels;*
2. *At each pixel location* (i,j), *add to the input pixel* $I(i,j)$*a weighted average of the previous errors* (*in some neighborhood* $N(i,j)$ *of* $I(i,j)$), *thus defining a modified input*

$$M(i,j) = I(i,j) + \sum_{(k,l)\in N(i,j)} w(i,j,k,l)E(k,l);$$

3. *Choose* $O(i,j)$ *as an element of V closest to* $M(i,j)$*;*
4. *Define the error* $E(i,j)$ *as* $M(i,j) - O(i,j)$.
5. *Reiterate until all pixels are processed.*

In practice, the error diffusion weights $w(i,j,k,l)$ are chosen to sum to 1 and are shift-invariant, i.e. they can be written as $w(i-k, j-l)$. In Definition 1, the output pixel is chosen as the output color closest (in the metric of the color space) to the modified input. This requirement can be relaxed to obtain the class of (not necessarily deterministic) generalized error diffusion algorithms[2]:

Definition 2 A generalized error diffusion algorithm is *defined by the following steps:*

1. *Choose an enumeration of the pixels;*
2. *At each pixel location* (i,j), *add to the input pixel* $I(i,j)$ *a weighted average of the previous errors* (*in some neighborhood* $N(i,j)$ *of* $I(i,j)$), *thus defining a modified input*

$$M(i,j) = I(i,j) + \sum_{(k,l)\in N(i,j)} w(i,j,k,l)E(k,l);$$

3. *Choose $O(i,j)$ depending on $M(i,j)$ such that $O(i,j) \in V$;*
4. *Define the error $E(i,j)$ as $M(i,j) - O(i,j)$.*
5. *Reiterate until all pixels are processed.*

Bounded Errors and BIBS Stability in Error Diffusion

The original error diffusion algorithm was developed for grayscale images where the input gamut S is the unit interval of gray values and $V = \{0,1\}$ are the two endpoints of the interval S, the colors black and white. A stability analysis for this case, also called scalar error diffusion, can be found in Ref. [3]. Until recently, little is known about the stability of error diffusion when the color spaces are higher dimensional spaces.* In particular, will the error grow arbitrarily large? The following result gives necessary and sufficient conditions for the existence of a generalized error diffusion algorithm such that the errors are bounded for color spaces in any dimension and arbitrarily large images[2]:

Theorem 1 *Suppose that V and N(i,j) are finite sets. S is contained in the convex hull of V if and only if there exists a generalized error diffusion algorithm such that all input images generate bounded errors.*

Thus the input gamut should be in the convex hull of the output colors to ensure bounded errors in (generalized) error diffusion. In order to prove one direction of Theorem 1, a generalized error diffusion algorithm is constructed in which errors, inputs and modified inputs are expressed as linear combinations of output vectors and all the operations are performed on the coefficients of these linear combinations. For simplicity in the sequel, pixels will be indexed by a single index i. Let $V = \{v_1, \ldots v_d\}$.

Algorithm 1

1. Choose an enumeration of the pixels;
2. Expand the input $I(i)$ as a (not necessarily unique) convex sum of points in V, i.e. $I(i) = \sum \mu_j v_j$;
3. At each pixel add to the input $I(i)$ the previous error $E(i-1)$, thus defining a modified input as $M(i) = I(i) + E(I-1)$;
4. Express $M(i) = \sum_{v_j \in V} \lambda_{ij} v_j$ as a linear combination of points in V as follows: if $E(I-1) = \sum \delta_j v_j$, then $\lambda_{ij} = \mu_j + \delta_j$.
5. Choose the output $O(i)$ as an element v_{j*} of V that satisfies $\lambda_{ij*} \geq \beta$ for some fixed β (the value of β can be chosen to be any number less than or equal to $1/d$ where d is the cardinality of V);
6. Define the error $E(i)$ of the current pixel as $M(i) - O(i)$ in the following way. If $O(i) = v_{j*}$, then $E(i) = \sum \kappa_j v_j$ where $\kappa_j = \lambda_{ij}$ when $j \neq j*$ and $\kappa_{j*} = \lambda_{ij*} - 1$.

The proof of Theorem 1 shows that $\kappa_j \in [\beta - 1, (1 - \beta)(d - 1)]$, i.e. there exists an arbitrarily large region for the error which is invariant for Algorithm 1 under all input images. Considering the error as a state of a dynamically system, this condition is sufficient for Bounded-Input-Bounded-State (BIBS) stability.[5] For simplicity, we have written Algorithm 1 for the case where the neighborhood $N(i,j)$ consists of a single pixel, i.e. the error from the previous pixel is added to the current pixel to create the modified input ($M(i) = I(i) + E(i - 1)$). We call this case *one-step* error diffusion. The boundedness result is also true for general sets of nonnegative weights $w(i,j,k,l)$ such that $\sum_{k,l} w(i,j,k,l) = 1$.[2]

When the output pixel $O(i)$ in Algorithm 1 is chosen as the element $v_{j*} \in V$ such that λ_{ij*} is the largest,† it is easy to show that the resulting algorithm is equivalent to mapping the convex hull of V to the probability simplex of dimension $d - 1$, performing error diffusion on the simplex and projecting the results back onto the color space. Let us refer to this as **Algorithm 1a**.

Algorithm 1 and 1a are generalized error diffusion algorithms. Is the error bounded for (classical) error diffusion as defined in Definition 1? This turns out to be a difficult question whose answer is affirmative. When the color space is one or two-dimensional, it is relatively easy to prove this.[6] For the higher dimensional case, this question is much more difficult to answer.[7] To answer this question, an bounded invariant region for the modified input (which implies a bounded error) is shown for the case of one-step error diffusion. By Theorem 5 in Ref. [2], this result is also true for arbitrary neighborhoods and weights.

On the other hand, generalized error diffusion as in Algorithm 1 performs better than error diffusion in some circumstances. For instance, error diffusion in color spaces such as Lab results in boundary artifacts due to large errors.[8] By using Algorithm 1a, this error can be made smaller.[9] Another example is the case where there are 3 output colors forming a triangle in 2-dimensional space. When one of the angles of the triangle is very close to 180°, it is easy to see that error diffusion can have arbitrarily large errors independent of the diameter of the triangle, whereas the errors in Algorithm 1 remain bounded with the bound depending only on the diameter.

Error diffusion and halftoning in general can be considered as an approximation problem. The goal is to approximate the input image I by the output image O such that the distance between I and O is small. A commonly used metric $d(O,I)$ to measure the perceptual difference between I and O

is to apply a lowpass filter to both images and calculate the norm of the difference between them. This can be expressed as $d(O,I) = \|L(O) - L(I)\|$ where L denotes the lowpass operator model of the human visual system.

So far, we have presented results which bound the error $E(i)$. Is there a relationship between E and the perceptual metric $d(O,I)$? It was shown in Ref. [2] that boundedness of E in error diffusion implies that the distance $d(O,I)$ is also bounded independent of the size of the image and that the more low pass the filter is, the smaller this distance is. This implies that at high enough resolution, $d(O,I)$ is small and (generalized) error diffusion generates a good halftone image. This validates the operation of error diffusion, even vector error diffusion in higher dimensional color spaces.

Applications to Scheduling

Consider the "Chairman Assignment Problem"[10,11]:

Let $X = \{x_1,...,x_d\}$ be a finite set, $d \geq 2$, and let μ_i be measures on X with $\mu_i(x_j) = \lambda_{ij} \geq 0$ for $1 \leq j \leq d$ and $\sum_{j=1}^{d} \lambda_{ij} = 1$ for all i. For an infinite sequence $\omega = (\omega_n)_{n=1}^{\infty}$ in X let $A(j,n,\omega)$ denote the number of occurrences of the elements x_j among the first n terms of ω and let $D(\omega) = \sup_{j,n} |A(j,n,\omega) - \sum_{k=1}^{n} \lambda_{kj}|$.

An application of this is a scheduling problem where d tasks compete for resources where at each time a single task is scheduled on the resources and each task requests a specific proportion of the resources.[12]

We consider algorithms which given μ_i, generate a sequence ω such that $D(\omega)$ is small. We distinguish between online algorithms where ω_i depends only on λ_{kj} for $k \leq i$ and *offline* algorithms. The terminology more appropriate for signal processing is causal algorithms versus *non-causal* algorithms. In Ref. [11], an optimal offline algorithm is presented for solving this problem.

For one-step error diffusion, the error $E(i)$ is equal to $\sum_{m<i} I(m) - O(m)$. It is easy to see that one-step error diffusion provides an online solution to the Chairman Assignment Problem. In particular, by considering the coefficients of the input pixel $I(i)$ in Algorithm 1 as the measure μ_i, we see that error diffusion gives a solution to the Chairman Assignment Problem with bounded discrepancy $D(\omega)$ (see Ref. [2] for more details).

Furthermore, one-step error diffusion on the probability simplex (Algorithm 1a) is optimal among all online algorithms for several scheduling problems including the Chairman Assignment Problem.[13]

This connection between error diffusion and scheduling can also benefit digital halftoning. For instance, the optimal offline algorithm presented in Ref. [11] can be cast into a *non-causal* halftoning algorithm which can generate smaller errors than error diffusion.[2]

Novel Applications in Image Processing

In the next section we present some applications of error diffusion to image processing besides image halftoning.

Imaging Viewable Under Different Conditions

Consider the scenario where a pixel behaves differently under different viewing conditions. The viewing conditions could be temperature, viewing angle, lighting condition, viewing apparatus (e.g. lens), etc. We would like to create printed (or displayed) images such that different images are seen depending on the viewing conditions. Digital halftoning, in particular error diffusion, provides a solution to this problem when the viewing conditions do not separate cleanly.

The main idea is to consider pixels living in the Cartesian product of color spaces. In other words, to each pixel P corresponds an n-tuple $(p_1,..., p_n)$ of points. Each point p_i is a vector in some color space C_i. Each p_i describes what the pixel P looks like under the i-th viewing condition, $1 \leq i \leq n$. We need enough output pixels which cover this product space in a nice way. The set of n input images is then converted into a single image in this product space. Performing halftoning in this product space results in an image which appears as different images depending on the viewing condition. An application of this algorithm to TN-mode LCD displays is presented in Ref. [14] where an image is constructed which when displayed on a LCD screen appears as different images depending on the viewing angle.

This idea of halftoning in product spaces is very useful as the following application indicate.

Data Hiding and Self-Repairing Images

Error diffusion has been found to be useful for image watermarking and data hiding.[15,16] In these methods, the embedded data or payload is relatively small compared to the source image. On the other hand, the algorithm in the previous application can be used for data hiding or image watermarking applications where an entire image can be embedded into another image.[17] A source image is considered as the first viewing condition and the watermark or hidden data is the second viewing condition. The way a pixel looks in the second viewing condition is chosen to minimize distortion in the watermarked image.

By embedding a scrambled version of the image into the same image, tampering to the image can be detected and reversed. This is accomplished by detecting where the tampering occurs and using the embedded image to correct the tampered parts of the image. One feature of this technique is that the reconstruction degrades gracefully as the tampering increases, i.e. the smaller the area tampered, the better the reconstruction is.

In Ref. [18], this data hiding method is generalized to embed m watermark images in n halftone images, where the extraction of the watermark images is performed via pixel-wise operations of the halftone images. For example, for the case $m = 1$, $n = 2$, the watermark image can be extracted by simply overlaying the 2 halftone images. Compared to other methods of watermarking with overlays which also uses error diffusion,[19] this method has the advantage of being able to extract high detail watermarks with high fidelity.

In these applications, suitable gamut reduction is usually needed to map the input space into the convex hull of the output pixels in order not to have large errors.

Imaging Scaling in DCT Domain

In the JPEG image compression format, the images are first divided into 8 by 8 blocks and transformed to the DCT domain and the DCT coefficients are processed to achieve compression. Since images are best stored and transmitted in compressed form and it is time consuming to compress and decompress an image, there is a need to be able to perform operations on images in compressed form. In Ref. [20] a method to scale images is proposed which operates in the DCT domain, eliminating the need to convert to the spatial domain to perform the scaling operations. The main idea is to have efficient algorithms which can perform scaling operations in the DCT domain for several fixed ratios of scaling. An arbitrary ratio can then be approximated by a sequence of fixed ratios such that on average the scaling is correct. This sequence is found via error diffusion. This algorithm can scale images faster than spatial algorithms. The approximation error causes distortion in the scaled image, but in many cases this distortion is visually unnoticeable.

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References

* Error diffusion in higher dimensional color spaces is sometimes referred to as vector error diffusion.[4]

† Since $\sum_j \lambda_{ij} = 1$, the largest λ_{ij} satisfies $\lambda_{ij} \geq 1/d \geq \beta$.

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Biography

Chai Wah Wu received the Ph. D. degree in electrical engineering from the University of California at Berkeley. He has published over 50 journal papers on various topics including digital halftoning, digital watermarking and nonlinear dynamics. He is a Fellow of IEEE and an associate editor of IEEE Transactions on Circuits and Systems-II. e-mail: chaiwahwu@ieee.org.