Explicit General Formulation of Color Matching Functions for Chromaticity Diagram Convexity and Its Application to Shape Structure Analysis

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Abstract

In this paper, the conditions required for convexity of chromaticity diagrams are discussed. We derive a general solution of color matching functions to satisfy convexity of chromaticity diagrams. Using the general solution, analysis related to the shape of color matching functions are performed which is impossible without the general solution. The results of this paper will contribute to a systematization of color matching functions from a view point of theoretical framework.

Introduction

In the field of color science,[1] there are several research issues that have not been covered to date, although they are considered to be of fundamental significance. The analysis of conditions related to color matching functions to satisfy the convexity of chromaticity diagrams is one of such issues.

In past studies, the convexity of chromaticity diagrams was investigated whether or not it is satisfied with given color matching functions,[2,3] but the general solution of color matching functions which satisfies convexity of the chromaticity diagrams has not yet been derived.

In this paper, we derive a general solution of color matching functions to satisfy convexity of chromaticity diagrams.[4] Using the general solution, an analysis related to the shape of color matching functions is performed which is impossible without the general solution. Using the general solution, a theorem related to the shape of color matching functions is provided. Related to the theorem, an example of non-convexity chromatic diagram is indicated. The results of this paper will contribute to a systematization of color matching functions from the theoretical point of view.

Basic Strategy

In the problem discussed in this study, when changes in the tangential direction in a two-dimensional coordinate system are considered, a function of $\tan(f(x,y))$ is inevitably included in differential equations describing the problem, and the equations become complicated. Accordingly, this problem has not been explicitly solved analytically. When equations are formulated for this problem with differential equations in a two-dimensional coordinate system, it is difficult to solve the differential equations; therefore, we describe the problem as being formulated by equivalent differential equations in a one-dimensional coordinate system. To realize this, the xy coordinate is rotated, and only the one-dimensional first-order and second-order differential equations in the y direction are considered for a local maximum point in the y direction. Rotation is performed around the origin $(0,0)$, and is performed with respect to the sampling points on the wavelength range λ, so that the tangential line at the wavelengths become horizontal by that rotation of amount $\psi(\lambda)$. Rotation is performed in the clockwise direction. With respect to $\lambda_c(1) < \lambda_c(2) \cdots < \lambda_c(n)$, the corresponding $\psi(\lambda_c(1))$, $\psi(\lambda_c(2))$ $\cdots$, $< \psi(\lambda_c(n))$ exist, where λ_c takes constant values on sampling points indexed 1 to n. As an initial state, if we assume that rotation is performed at the position where the tangential line for $\lambda_c(1)$ becomes horizontal, the relationship of $\psi(\lambda_c(1)) < \psi(\lambda_c(2)) < \cdots < \psi(\lambda_c(n))$ holds under the prerequisite of establishment of convexity.

Figure 1 shows a process in which ψ has the minus value (a), and ψ has the plus value (b). On xy axis coincidence, $\psi = 0$.

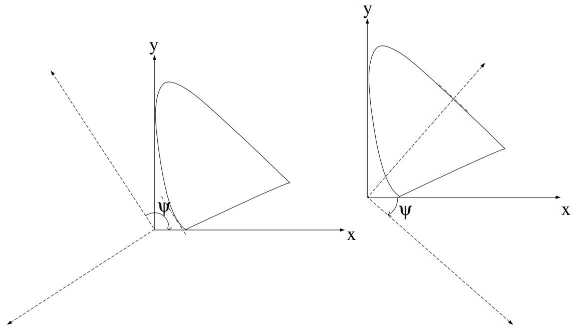

Figure 1(a) (left), (b) (right)

Differential Equations and Solution

$\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ are assumed to be color matching functions. For simplicity, we assume that the following equation holds.

$$u(\lambda) = \bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda) . \tag{1}$$

The coordinate values (x^*, y^*) to realize the local maximum point of the chromaticity diagram by rotation and displacement are obtained as

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \begin{bmatrix} \dfrac{\bar{x}(\lambda)}{u(\lambda)} \\ \dfrac{\bar{y}(\lambda)}{u(\lambda)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{r_1\bar{x}(\lambda) + r_2\bar{y}(\lambda)}{u(\lambda)} \\ \dfrac{r_3\bar{x}(\lambda) + r_4\bar{y}(\lambda)}{u(\lambda)} \end{bmatrix} , \tag{2}$$

where

$$\begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} = \begin{bmatrix} \cos\psi(\lambda_c) & -\sin\psi(\lambda_c) \\ \sin\psi(\lambda_c) & \cos\psi(\lambda_c) \end{bmatrix} \text{: rotation matrix,}$$

$\psi(\lambda_c)$: the rotational angle at which a tangential line at sampling points on the wavelength range λ becomes horizontal, where λ_c takes constant values on sampling points,

With respect to y^* in eq. (2), let us consider conditions with which the first-order differential becomes 0 and upward convexity is achieved.

Condition With Which First-Order Differential Becomes 0

For simplicity, let assume

$$v(\lambda) = r_3\bar{x}(\lambda) + r_4\bar{y}(\lambda) . \tag{3}$$

Then, the following equation holds.

$$\frac{dy^*}{d\lambda} = \frac{v'(\lambda)u(\lambda) - v(\lambda)u'(\lambda)}{u^2(\lambda)} = 0 . \tag{4}$$

Due to the condition that the numerator of Eq.(4) = 0, the following equations hold.

$$v'(\lambda)u(\lambda) = v(\lambda)u'(\lambda) , \tag{5}$$

$$\frac{v'(\lambda)}{v(\lambda)} = \frac{u'(\lambda)}{u(\lambda)} . \tag{6}$$

By integrating both sides of Eq.(6), the following equations are derived.

$$\log(v(\lambda)) = \log(u(\lambda)) + K_1 \tag{7}$$

$$e^{K_1} = \frac{v(\lambda)}{u(\lambda)} > 0 \tag{8}$$

where

K_1 : Integral constant (arbitrary constant).

Since $0 < u(\lambda)$,

$$0 < v(\lambda) = r_3\bar{x}(\lambda) + r_4\bar{y}(\lambda) , \tag{9}$$

holds.

Condition With Which Upward Convexity is Realized

The following condition to realize upper convexity is provided.

$$\begin{aligned} \frac{d^2y^*}{(d\lambda)^2} &= \{(v'(\lambda)u(\lambda) - v(\lambda)u'(\lambda))' u^2(\lambda) \\ &- 2(v'(\lambda)u(\lambda) - v(\lambda)u'(\lambda))\}u(\lambda)u'(\lambda)/u^4(\lambda) \\ &< 0 . \end{aligned} \tag{10}$$

Because of Eq. (5), the second term of the numerator in Eq.(10) is 0. The first term of the numerator in Eq.(10) is

$$(v'(\lambda)u(\lambda) - v(\lambda)u'(\lambda))' u^2(\lambda) = (v''(\lambda)u(\lambda) - v(\lambda)u''(\lambda))u^2(\lambda) . \tag{11}$$

Since the following equation holds in Eq.(11) due to the condition of Eq.(10),

$$0 > (v''(\lambda)u(\lambda) - v(\lambda)u''(\lambda)) , \tag{12}$$

we set

$$(v''(\lambda)u(\lambda) - v(\lambda)u''(\lambda)) = K_2(\lambda) . \tag{13}$$

where, $K_2(\lambda)$ is a parameter to express Eq.(12) as the equality Eq.(13), which satisfies $0 > K_2(\lambda)$.

General Solution Regarding $\bar{x}(\lambda)$

Equation (13) is expressed as follows with regard to $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$.

$$\left(r_3\bar{y}(\lambda)+r_3\bar{z}(\lambda)-r_4\bar{y}(\lambda)\right)\bar{x}''(\lambda)-\left(r_3\bar{y}''(\lambda)+r_3\bar{z}''(\lambda)-r_4\bar{y}''(\lambda)\right)\bar{x}(\lambda)$$
$$=K_2(\lambda)-\left(r_4\bar{y}''(\lambda)\bar{z}(\lambda)-r_4\bar{y}(\lambda)\bar{z}''(\lambda)\right)=S(\lambda)\ , \qquad (14)$$

where

$$S(\lambda)=K_2(\lambda)-\left(r_4\bar{y}''(\lambda)\bar{z}(\lambda)-r_4\bar{y}(\lambda)\bar{z}''(\lambda)\right). \qquad (15)$$

To simplify this equation, assuming that

$$Q(\lambda)=r_3\bar{y}(\lambda)+r_3\bar{z}(\lambda)-r_4\bar{y}(\lambda)\ , \qquad (16)$$

Eq. (14) can be expressed as

$$Q(\lambda)\bar{x}''(\lambda)-Q''(\lambda)\bar{x}(\lambda)=S(\lambda)\ , \qquad (17)$$

which is the second-order differential equation to be solved.

First, a particular solution for Eq.(17) is obtained. The particular solution is a solution of the following equation.

$$Q(\lambda)\bar{x}''(\lambda)-Q''(\lambda)\bar{x}(\lambda)=0\ . \qquad (18)$$

The particular solution $\bar{x}_s(\lambda)$ can be easily obtained in the following form.

$$\bar{x}_s(\lambda)=Q(\lambda)=r_3\bar{y}(\lambda)+r_3\bar{z}(\lambda)-r_4\bar{y}(\lambda)\ (\neq 0)\ . \qquad (19)$$

The non-zero condition in Eq.(19) is the condition for the integral operations following.

Next, a general solution is obtained using d'Alembert's method[5], in which the following equation (Eq.(20)) that are based on the particular solution are designated to be the solution of the differential equation.

$$H(\lambda)=h(\lambda)\bar{x}_s(\lambda)\ , \qquad (20)$$

$$H''(\lambda)=h''(\lambda)\bar{x}_s(\lambda)+2h'(\lambda)\bar{x}_s'(\lambda)+h(\lambda)\bar{x}_s''(\lambda). \qquad (21)$$

By substituting $H(\lambda)$ and $H''(\lambda)$ into Eq.(17), the following equation is derived.

$$\bar{x}_s(\lambda)\left(h''(\lambda)\bar{x}_s(\lambda)+2h'(\lambda)\bar{x}_s'(\lambda)+h(\lambda)\bar{x}_s''(\lambda)\right)-\bar{x}_s''(\lambda)h(\lambda)\bar{x}_s(\lambda)=S(\lambda) \qquad (22)$$

By rearranging Eq.(22),

$$h''(\lambda)+2\frac{\bar{x}_s'(\lambda)}{\bar{x}_s(\lambda)}h'(\lambda)=\frac{S(\lambda)}{\bar{x}_s^2(\lambda)}. \qquad (23)$$

If we assume that $h'(\lambda)=h_1(\lambda)$ in Eq.(23), Eq.(23) becomes a first-order differential equation with respect to $h_1(\lambda)$, as follows.

$$h_1'(\lambda)+2\frac{\bar{x}_s'(\lambda)}{\bar{x}_s(\lambda)}h_1(\lambda)=\frac{S(\lambda)}{\bar{x}_s^2(\lambda)}. \qquad (24)$$

This first-order differential equation with respect to $h_1(\lambda)$ can be solved based on a general method; first, we obtain solutions for the following linear homogenous differential equation.

$$h_1'(\lambda)+2\frac{\bar{x}_s'(\lambda)}{\bar{x}_s(\lambda)}h_1(\lambda)=0\ . \qquad (25)$$

A solution for Eq.(25) can be obtained as

$$h_1(\lambda)=\exp\left(-\int 2\frac{\bar{x}_s'(\lambda)}{\bar{x}_s(\lambda)}d\lambda\right)=\frac{1}{\bar{x}_s^2(\lambda)}\ . \qquad (26)$$

Solutions for nonlinear homogenous differential equations can be obtained using d'Alembert's formula[5] as

$$h_1(\lambda)=K_3\frac{1}{\bar{x}_s^2(\lambda)}+\frac{1}{\bar{x}_s^2(\lambda)}\int S(\lambda)d\lambda\ . \qquad (27)$$

The integration for both sides of Eq.(27) derives the following equation.

$$h(\lambda)=K_3\int\frac{1}{\bar{x}_s^2(\lambda)}d\lambda+\int\frac{1}{\bar{x}_s^2(\lambda)}\int S(\lambda)d\lambda d\lambda+K_4\ , \qquad (28)$$

where

K_3, K_4 : Integral constant.

The sum of the general solution $H(\lambda)=h(\lambda)\bar{x}_s(\lambda)$ and the particular solution of Eq.(19) is the solution $\bar{x}(\lambda)$ for the differential equation (17).

$$\bar{x}(\lambda)=\bar{x}_s(\lambda)\left(1+h(\lambda)\right). \qquad (29)$$

This is a general function form of $\bar{x}(\lambda)$.

The Relationship Between λ and ψ

For solutions that are discretized and obtained with respect to the wavelength $\lambda_c(1)<\lambda_c(2)\cdots<\lambda_c(n)$, corresponding $\psi(\lambda_c(1)),\psi(\lambda_c(2)),\cdots,\psi(\lambda_c(n))$ exist; and the relationship of $\psi_{min}<\psi(\lambda_c(1))<\psi(\lambda_c(2))<\cdots<\psi(\lambda_c(n))<\psi_{max}$ holds under the prerequisite of convexity, where ψ_{min} and ψ_{max} are the minimum and the maximum angles, respectively. There exists sufficient large n with which the relations of $|\lambda_c(i-1)-\lambda_c(i)|<\eta_1$ and $|\psi(\lambda_c(i-1))-\psi(\lambda_c(i))|<\eta_2$ are satisfied for given values of η_1, η_2. All of relations between λ and ψ which hold this relationships in the solution framework of Eq.(29) are accepted for the solutions. It is assumed that, n takes sufficient large finite value for approximation.

General Solution Regarding $\bar{y}(\lambda)$

The solution is derived in the same way as $\bar{x}(\lambda)$ follows.

$$\bar{y}(\lambda)=\bar{y}_s(\lambda)\left(1+h(\lambda)\right)\ , \qquad (30)$$

where

$$\bar{y}_s(\lambda)=Q(\lambda)=r_4\bar{x}(\lambda)+r_4\bar{z}(\lambda)-r_3\bar{x}(\lambda)\ (\neq 0)\ , \qquad (31)$$

$$h(\lambda)=K_3\int\frac{1}{\bar{y}_s^2(\lambda)}d\lambda+\int\frac{1}{\bar{y}_s^2(\lambda)}\int S(\lambda)d\lambda d\lambda+K_4\ , \qquad (32)$$

$$S(\lambda)=K_2(\lambda)+\left(r_3\bar{x}(\lambda)\bar{z}''(\lambda)-r_3\bar{x}''(\lambda)\bar{z}(\lambda)\right)\ , \qquad (33)$$

$K_2(\lambda)$, K_3, K_4 : Identical definition in the section of $\bar{x}(\lambda)$.

The general solutions of $\bar{x}(\lambda)$ and $\bar{y}(\lambda)$ are constrained by the condition of Eq.(9).

General Solution Regarding $\bar{z}(\lambda)$

$$\bar{z}(\lambda) = \bar{z}_s(\lambda)(1 + h(\lambda)), \tag{34}$$

where

$$\bar{z}_s(\lambda) = Q(\lambda) = -r_3\bar{x}(\lambda) - r_4\bar{y}(\lambda)\ (\neq 0), \tag{35}$$

$$h(\lambda) = K_3 \int \frac{1}{\bar{z}_s^2(\lambda)} d\lambda + \int \frac{1}{\bar{z}_s^2(\lambda)} \int S(\lambda) d\lambda d\lambda + K_4, \tag{36}$$

$$S(\lambda) = K_2(\lambda) + \left(r_4\bar{x}''(\lambda)\bar{y}(\lambda) + r_3\bar{x}(\lambda)\bar{y}''(\lambda) - r_3\bar{x}''(\lambda)\bar{y}(\lambda) - r_4\bar{y}''(\lambda)\bar{x}(\lambda)\right), \tag{37}$$

$K_2(\lambda)$, K_3, K_4 : Identical definition in the section of $\bar{x}(\lambda)$.

Shape Structure Analysis Using the General Solution

In general, it has been recognized that color matching functions are basically smooth. Using the general solution derived in the previous chapter, an analysis related to the shape of color matching functions is performed which is impossible without the general solution.

[Assumption]

Color matching functions are assumed to be continuous functions.

[Theorem]

For color matching functions satisfying convexity of its chromaticity diagram, if there is at least one non-smooth function in $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$, the rest functions should also be non-smooth functions on the same wavelength (case I). The other case satisfying convexity is that all of color matching functions are smooth functions (case II). Where, non-smooth implies that there is at least one curve point on which the first-order difference is discontinuous.

Proof

Generally, the following characteristics have been proved.

Characteristic 1

Operations between continuous functions result in a continuous function, and operations between smooth functions result in a smooth function.

Characteristic 2

Operations between a non-smooth function and a smooth function result in a non-smooth function.

Let assume $\bar{x}(\lambda)$ to be a non-smooth function.

In Eq.(32), $1/\bar{y}_s^2(\lambda)$ in the first term is a continuous function from Characteristic 1, because $\bar{x}(\lambda)$, $\bar{z}(\lambda)$, r_3, r_4 are continuous functions. $\int S(\lambda) d\lambda$ in the second term is a continuous function because of the integral operation, and the multiplication between $1/\bar{y}_s^2(\lambda)$ and $\int S(\lambda) d\lambda$ also results in a continuous function from Characteristic 1. In the calculation of Eq.(32), the integral calculations make $h(\lambda)$ a smooth function. This can be confirmed from that the first-order differential of $h(\lambda)$ becomes a continuous function based on the considerations described. In Eq.(30), $(1 + h(\lambda))$ is also a smooth function from Characteristic 1. In Eq.(30), $\bar{y}_s(\lambda) = r_4\bar{x}(\lambda) + r_4\bar{z}(\lambda) - r_3\bar{x}(\lambda)$ becomes a non-smooth function from the assumption and Characteristic 2, and the multiplication between $\bar{y}_s(\lambda)$ and $(1 + h(\lambda))$ (Eq.(30)) becomes a non-smooth function from Characteristic 2. In the same way, Eq.(34) becomes a non-smooth function. Here, it is proved that if $\bar{x}(\lambda)$ is a non-smooth function, $\bar{y}(\lambda)$ and $\bar{z}(\lambda)$ should be non-smooth functions.

In the case of that $\bar{y}(\lambda)$ is non-smooth, or $\bar{z}(\lambda)$ is non-smooth, the same proofs can be provided.

Based on Eqs.(29),(30),(34), the other case that all of color matching functions are smooth functions can be proved easily from Characteristic 1.

The theorem indicates that there are two cases that all of color matching functions are smooth functions, or all of color matching functions are non-smooth functions under the assumption of convexity of its chromaticity diagram. This is a new knowledge related to color matching functions.

Examples Related to Theorem

Figure 2 is an example of color matching functions for non-convexity, and Figures 3 is corresponding chromaticity diagram, respectively. In figure 2, a part of color matching functions which does not satisfy Theorem exists near 440nm and the result of Figure 3 has a non-convex part. The color matching functions of Figure 2 is sampled in 1nm wavelength and can be seem to be the original and the non-smooth parts are included in only one of the original color matching functions of $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$. The non-convexity characteristics are included in the original color matching functions. This case does not correspond to one of the two cases for convexity in Theorem. Where the original implies that the sampling interval is sufficiently small in practice.

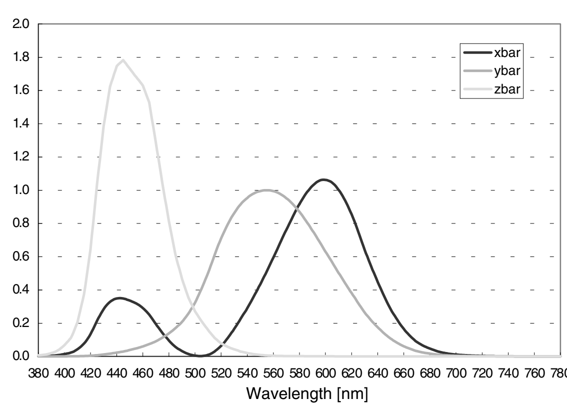

Figure 2. Example of color matching functions for non-convexity.

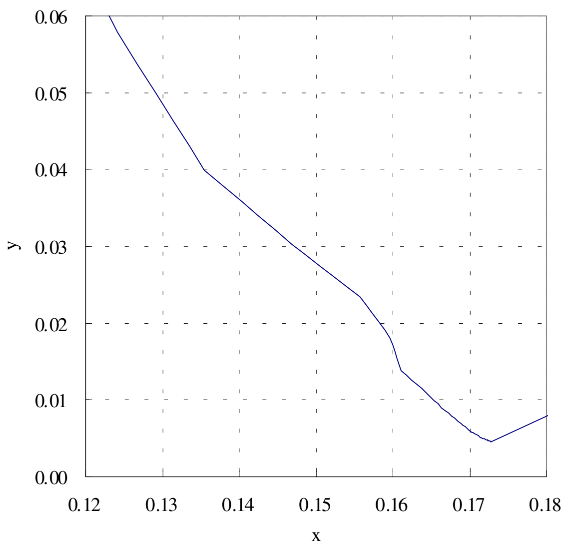

Figure 3. Chromaticity diagram (Fig.2 color matching functions).

Also it has been confirmed that the color matching functions of Figure 2 does not satisfy Eqs.(29)(30)(34).

The considerations described above are from Theorem derived from the general solution. The existence of the general solution enabled the analysis related to the numerical example.

Conclusions

In this paper, we derived a general solution of color matching functions which satisfy the convexity of the chromaticity diagram. The solution is an original result, and applied to shape structure analysis related to color matching functions. A theorem derived from the general solution indicated that there are two cases that all of color matching functions are smooth functions, or all of color matching functions are non-smooth functions under the assumption of convexity of its chromaticity diagram. This is a new knowledge related to color matching functions. Numerical example was corresponded to the theorem included. Without the general solution, it was impossible to derive the new knowledge, and it will be a strong tool for various problems related to color matching functions.

Hereafter, we will apply the general solution to other problems unsolved.

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Biography

Nobuhito Matsushiro received his Ph.D degree related to information theory from University of Electro. Communications, Tokyo, Japan, in 1996. Now, he is a visiting scientist at Munsell Color Science Laboratory, Center for Imaging Science, Rochester Institute of Technology, and he work for Oki Data Corporation, Japan. His recent research interest lies in mathematical approach to color science and color engineering.