Color Gamut Boundaries in CIELAB Space

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Abstract

Color gamut boundaries in CIELAB space are determined for two CIE standard observers, CIE 1931 2° and CIE 1964 10°, under two illuminants, D_{50} and D_{65}, using spectral and optimal stimuli. Computational methods and formulas based on Grassmann's law are given. Two-dimensional projection of the computed color gamut provides the boundary for all possible stimuli, real and imaginary, and is the closest analogue to the chromaticity diagram of the CIEXYZ or CIELUV space. We also discuss reasons of no chromaticity diagram in CIELAB space. This computation leads to the derivation of the most saturated primary colors for block dyes. Two-dimensional gamut projections of several sets of block dyes are then computed under conditions of the CIE 1931 color match functions and illuminants D_{50} and D_{65}. The validity of the Grassmann's law is examined for additive and subtractive color mixing of block dyes. Finally, color gamut boundaries of two real devices are presented for comparisons.

Introduction

CIEXYZ space gives a well-defined color gamut boundary that is enclosed by spectral stimuli. The gamut boundary is computed from color matching functions (CMF), $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, and $\bar{z}(\lambda)$ of the CIE 1931 2° or CIE 1964 10° observer by using Eq. (1) to give chromaticity coordinates of spectral stimuli.[1]

$$x(\lambda) = \bar{x}(\lambda) / [\, \bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda) \,], \tag{1.1}$$

$$y(\lambda) = \bar{y}(\lambda) / [\, \bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda) \,], \tag{1.2}$$

where x and y are the chromaticity coordinates and λ is the wavelength. The chromaticity diagram is the plot of x vs. y. CIE 1931 CMF gives a slightly larger gamut size than CIE 1964 CMF in green, blue, and purple regions.

Similarly, CIELUV has a well-defined gamut boundary that is based on CIE 1976 uniform-chromaticity-scale (UCS) u' and v'.

$$u'(\lambda) = 4\,\bar{x}(\lambda) / [\, \bar{x}(\lambda) + 15\,\bar{y}(\lambda) + 3\,\bar{z}(\lambda)], \tag{2.1}$$

$$v'(\lambda) = 9\,\bar{y}(\lambda) / [\, \bar{x}(\lambda) + 15\,\bar{y}(\lambda) + 3\,\bar{z}(\lambda)]. \tag{2.2}$$

Again, CIE 1931 CMF is larger than CIE 1964 CMF in CIELUV, where the gamut size difference between these two observers seems to become wider in this space.

Another frequently used color space CIELAB is defined in Eq. (3).

$$L^* = 116\,(Y/Y_n)^{1/3} - 16, \tag{3.1}$$

$$a^* = 500\,[\,(X/X_n)^{1/3} - (Y/Y_n)^{1/3}\,], \tag{3.2}$$

$$b^* = 200\,[\,(Y/Y_n)^{1/3} - (Z/Z_n)^{1/3}\,], \tag{3.3}$$

where X, Y, and Z are the tristimulus values of the object and X_n, Y_n, and Z_n are the tristimulus values of the illuminant. Unlike CIEXYZ and CIELUV spaces, CIELAB does not have a chromaticity diagram. As pointed out by Bartleson, this is because values of a^* and b^* depend on L^*.[2] But, this does not mean that there is no gamut boundary for CIELAB. As early as 1975, Judd and Wyszecki already gave a graphic outer boundary of the CIELAB space with respect to illuminant D_{65} and CIE 1964 observer.[3] Although the detail was not revealed on how this graph was generated, they did mention the use of optimal color stimuli. Optimal color stimulus is an imaginary stimulus, having two or more spectral components (usually, no more than 3 wavelengths) with a unit reflectance at the specified wavelength and zero elsewhere. No real object surfaces can have this kind of abrupt reflectance curves.

Because of lacking details of the CIELAB gamut boundary, the purpose of this paper is trying to fill this gap by deriving color gamut boundaries for CIELAB space. Along the way, we also try to find the gamut boundary for ideal block dyes, determine the optimal spectra for primary colors, and check the validity of the Grassmann's law for color mixing.

Color Gamut Boundaries in CIELAB Space

In an effort to reproduce the graph given by Judd and Wyszecki, we perform a series of computations using spectral and optimal stimuli of CIE standard observers. First, we compute tristimulus values of the spectral stimuli, then we compute tristimulus values of all possible two-component optimal stimuli. These tristimulus values are converted to L^*, a^*, and b^* via Eq. (3). For computing gamut boundary in CIELAB, tristimulus values of the object in Eq. (3) are replaced by a CMF at a given

wavelength λ and tristimulus values of the illuminant is normalized to Y_n=1.

$$L^* = 116\,[\bar{y}(\lambda) / Y_n]^{1/3} - 16\,, \tag{4.1}$$

$$a^* = 500\,\{[\bar{x}(\lambda) / X_n]^{1/3} - [\bar{y}(\lambda) / Y_n]^{1/3}\}\,, \tag{4.2}$$

$$b^* = 200\,\{[\bar{y}(\lambda) / Y_n]^{1/3} - [\bar{z}(\lambda) / Z_n]^{1/3}\}\,, \tag{4.3}$$

Two-dimensional projection of CIE spectral stimuli under either D_{50} or D_{65} in CIELAB shows that stimuli at both high- and low-wavelength ends locate near the origin of the chromaticity diagram. These results give a concave area in the positive a^* quadrants encompassed color regions from red through magenta to purple. This is because stimuli are weak at the both ends of the CMF, having a very small brightness and colorfulness. Thus, they are perceived as if they were achromatic hues.

It is a known fact that a single spectral stimulus does not give saturated magenta colors. Magenta colors require at least two spectral stimuli in proper wavelengths with respect to each other. Therefore, we start by computing all possible optimal stimuli of two spectral wavelengths in the range from 390 nm to 750 nm. This is achieved by fixing one stimulus at a wavelength λ_1 and moving the other stimulus across the whole spectrum at a 5 nm interval, starting from $\lambda_2=\lambda_1+5$ nm until it reaches the other end of the spectrum. The first stimulus is then moved to the next wavelength and the second stimulus again is scanned across the remaining spectrum at the 5 nm interval. This procedure is repeated until both stimuli reach the other end. Note that the 5 nm interval is an arbitrary choice. One can reduce the interval to get more precise wavelength location or one can widen the interval to save computation cost with some sacrifice of the accuracy.

We employ Grassmann's additivity law given in Eq. (5) with normalization (actually, this is the averaging because Y_n is normalized to 1) to compute tristimulus values, X, Y, and Z of two stimuli.

$$X = [\,\bar{x}(\lambda_1) + \bar{x}(\lambda_2)] / 2\,, \tag{5.1}$$

$$Y = [\,\bar{y}(\lambda_1) + \bar{y}(\lambda_2)] / 2\,, \tag{5.2}$$

$$Z = [\,\bar{z}(\lambda_1) + \bar{z}(\lambda_2)] / 2\,. \tag{5.3}$$

These tristimulus values are then plugged into Eq. (3) to obtain CIELAB values. With additional data form two-component optimal stimuli, we fill the concave area in the positive a^* quadrants when only spectral stimuli are plotted. Connecting outer-most colors produced by optimal and spectral stimuli, we obtain the boundary of the CIELAB space with respect to the illuminant used. For D_{65}, the resulting gamut looks pretty close to the outer boundary given by Judd and Wyszecki.[1] We also obtain results for other combinations of illuminant and standard observer. For CIE 1931 observer under illuminants D_{50} and D_{65}, D_{65} gives a substantial larger area in green, cyan, blue, and purple regions, whereas D_{50} gives a slightly larger area in magenta and red regions. CIE 1964 observer under D_{50} and D_{65} gives a similar trend as in the case of the CIE 1931 observer. Under the same illuminant, CIE 1964 observer is slightly larger than CIE 1931 observer in cyan and blue regions, whereas CIE 1931 observer is larger in green, red, magenta, and purple regions. Generally speaking, the color gamut difference is small with respect to the difference in standard observer, but it is large with respect to the difference in illuminant used.

Color Gamut Boundary of Block Dyes

Block dyes are similar to optimal stimuli in that they are also imaginary colorants, having a uniform spectrum of a unit reflectance within specified wavelength ranges and zero elsewhere. Again, no real colorants process this kind of spectra. We generate ideal block dyes by selecting a bandwidth and moving it from one end of the visible spectrum to the other end. We also produce block dyes with complementary spectrum $P_c(\lambda)$; that is, unit reflectance subtracts a block dye spectrum $P(\lambda)$ across the whole visible region,

$$P_c(\lambda) = 1 - P(\lambda)\,. \tag{6}$$

We start at a bandwidth of 10 nm at one end and move it at a 10 nm interval for generating a whole series of block dyes. We then increase the bandwidth by 10 nm to generate next set of block dyes. This procedure is repeated until the bandwidth reaches 220 nm. We compute tristimulus values of these block dyes by using the CIE standard formulation given in Eq. (7).

$$X = k \sum P(\lambda)\, I(\lambda)\, \bar{x}(\lambda)\, \Delta\lambda\,, \tag{7.1}$$

$$Y = k \sum P(\lambda)\, I(\lambda)\, \bar{y}(\lambda)\, \Delta\lambda\,, \tag{7.2}$$

$$Z = k \sum P(\lambda)\, I(\lambda)\, \bar{z}(\lambda)\, \Delta\lambda\,, \tag{7.3}$$

$$k = 1 / [\,\sum I(\lambda)\, \bar{y}(\lambda)\, \Delta\lambda\,]\,, \tag{7.4}$$

where $P(\lambda)$ is the spectrum of an object and $I(\lambda)$ is the spectrum of an illuminant. We then compute CIELAB values using Eq. (3) to derive the gamut boundary for block dyes. The shape of the two-dimensional gamut boundary of block dyes is pretty similar to the corresponding boundary of spectral and optimal stimuli with a much smaller gamut size, whereas the three-dimensional gamut shows some similarity to the boundary of object-color stimuli given by Judd and Wyszecki.[1]

Ideal Primary Colors of Block Dyes

Outer boundary of block dyes can be used to find ideal primary colorants with respect to color strength. The most saturated red has a spectrum with wavelength onset around 600 nm and cutoff around 680 nm. It is not sensitive to broaden the spectrum at long wavelength side because

CMF is very weak at the red end. On the short wavelength side, we can broaden the spectrum as low as 550 nm, but hue is gradually shifted toward orange.

The most saturated green has a spectrum with wavelength between 500 to 560 nm. There are various shades of greens; hue shifts toward cyan if we broaden short wavelength end of the spectrum and shifts toward yellow if we broaden long wavelength end.

The most saturated blue has a spectrum of [400, 470] nm. By shifting a blue spectrum with a fixed bandwidth to higher wavelengths, hue gradually changes to cyan. We still get blue color when the long wavelength ends around 510 nm. Similar to shifting, hue gradually changes to cyan by broadening the spectrum at long wavelength side. But, broadening at short wavelength side is not sensitive to the hue shift. This is because CMF is very weak at the blue end. We still get blue color with a bandwidth of [390, 520] nm.

The most saturated cyan has a spectrum of [400, 560] nm. By broadening cyan spectrum at long wavelength side, hue shifts toward green. It is still considered as cyan with a bandwidth of [400, 610] nm. The change at short wavelength side is not sensitive to the hue shift. However, the lower bound should be no higher than 430 nm, otherwise the hue will be too greenish. To produce wider range of greens, the range may be extended to [390, 580] nm.

The most saturated magenta has a spectrum of [400, 460] and [600, 700] nm. By extending short wavelength side of the red component, hue shifts from purple/blue through magenta toward red. By broadening long wavelength side of the blue component, hue remains as magenta color with a diminished chroma. By reducing blue component, hue shifts toward red.

The most saturated yellow has a spectrum of [540, 620] nm. By broadening yellow spectrum at long wavelength side, hue becomes warmer (toward red). It is still considered as yellow with [540, 700] nm bandwidth. By broadening short wavelength side, hue becomes cooler (toward green). We still get yellow with a bandwidth of [490, 700] nm. To make a broad range of greens, especially the darker bluish-greens, yellow spectrum should be extended to lower wavelengths. Thus, the cool yellows are preferred.

Interestingly, chromaticity coordinates of RGB block dyes are pretty close to certain standard RGB primaries such as NTSC/RGB or sRGB. This seems to imply that behind a simple chromaticity representation of the primary, there is an associated block dye spectrum. This association is not surprising because many RGB primaries are spectral stimuli such as CIE primaries. The chromaticity coordinates of a primary can be matched by a block dye if we finely tune the spectrum on a wavelength-by-wavelength basis.

Additive Color Mixing of Block Dyes

We compute the color mixing of the most saturated RGB block dyes (RGB set #1) to obtain the color gamut. For comparison purpose, we perform the same additive mixing on a set of RGB block dyes with broader spectra, having red at [600, 720] nm, green at [500, 600] nm, blue at [390, 500] nm (RGB set #2. In addition, we also compute the additive mixing of a set of CMY block dyes with cyan at [390, 590] nm, magenta at [390, 490] and [590, 750] nm, and yellow at [490, 750] nm (CMY set #2).

Equation (8) gives the formula of the additive mixing; it is a linear combination of block dyes at a given wavelength.

$$P(\lambda) = w_r P_r(\lambda) + w_g P_g(\lambda) + w_b P_b(\lambda), \quad (8)$$

where w_r, w_g, and w_b are the weights or concentrations of red, green, and blue components, respectively, in the mixture, and P_r, P_g, and P_b are the respective reflectance of red, green, and blue dyes. The computation indicates that RGB set #1 indeed has the largest color gamut. Two-color mixtures lie perfectly in lines of a triangle, indicating that the additive mixing obeys Grassmann's law, regardless the type of block dyes whether RGB or CMY. Converting to CIELAB, the data no longer lie in a straight line, indicating that the common practice of connecting R, Y, G, C, B, and M primary colors in straight lines is not a good way to define color gamut size.

Subtractive Color Mixing of Block Dyes

We also compute color gamut formed by the most saturated CMY block dyes (CMY set #1) and CMY set #2 in subtractive mixing. Equation (9) gives the formula of the subtractive color mixing at a given wavelength.

$$P(\lambda) = \exp\text{-}\{w_c \log[P_c(\lambda)] + w_m \log[P_m(\lambda)] + w_y \log[P_y(\lambda)]\}, \quad (9)$$

where w_c, w_m, and w_y are the weights of cyan, magenta, and yellow components, respectively, in the mixture, and P_c, P_m, and P_y are the respective reflectance of cyan, magenta, and yellow dyes. The reflectance is converted to optical density by taking the logarithm base 10. This logarithm conversion poses a problem for block dyes because they have zero reflectance at many wavelengths. Mathematically, the logarithm of zero is undefined. Therefore, a density value must be artificially assigned to zero reflectance. We choose to give a maximum density of 4, which is high enough for most, if not all, practical applications. The individual densities of subtractive primaries are weighed and combined to give the density of the mixture at a given wavelength. Finally, the density is converted back to reflectance for computing tristimulus and CIELAB values via Eq. (3).

Based on Eq. (9) with a maximum density of 4, we compute the color gamut of CMY block dye set #1 and #2 under D_{65} in CIEXYZ and CIELAB spaces. In CIEXYZ space, CMY set #2 forms a nearly perfect triangle with vertices located in R, G, and B regions. All two-color mixtures lie in lines of the triangle. CMY set #1 does not give a triangle, instead a pentagon, where two-color mixtures fall in lines connecting R, G, C, B, and M vertices. The resulting color gamut resembles closely to the chromaticity diagram of primaries employed in

Donaldson's six-primary colorimeter.[4,5] The difference between CMY set #1 and #2 may be attributed to the fact that the complementary spectra of CMY set #1 are overlapped, whereas complementary spectra of CMY set #2 form a continuous white spectrum with no overlapping. These computations indicate that Grassmann's additivity is followed, if not strictly obeyed, for block dyes even in the subtractive mixing. By transforming to CIELAB, both dye sets give a rather curved boundary with six vertices. These results reaffirm that the straight-line connection of six primaries will give a misled color gamut in CIELAB space.

In addition, we also compare the computed block dye color gamut with two measured real inks. The real inks give a pretty straight hexagon shape in CIELAB, indicating that the practice of connecting six primaries in straight lines is a reasonable approximation. They give a slightly curved polygon in CIEXYZ, indicating that they do not obey Grassmann's law, but they can be modeled by other color mixing theories that take the complex phenomena of absorption, reflection, and transmission into account.[6]

Remarks

The detail formulation and computation of the color gamut boundary in CIELAB space are given in this paper. By employing Eqs. (4) and (5), the computation indicates that there is indeed a gamut boundary in CIELAB space using spectral and optimal stimuli. However, unlike chromaticity diagrams of CIEXYZ and CIELUV, the size of the gamut depends on the illuminant used. The illuminant has a significant influence on the size of the gamut boundary in CIELAB space, whereas the standard observer only has a minor effect.

Spectral and optimal stimuli obey Grassmann's law of the color mixing. Likewise, block dyes, regardless RGB or CMY set, obey Grassmann's law of the additive color mixing. Surprisingly, subtractive block dyes in subtractive color mixing follow Grassmann's law closely, whereas the real inks do not.

From this computation, the common practice of connecting six primaries in CIELAB space to give a projected two-dimensional color gamut is not a strictly correct way of defining a color gamut. However, it is a pretty close approximation for real colorants. To have a better definition of the color gamut, more two-color mixtures should be measured and plotted in CIELAB space.

References

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Biography

Henry R. Kang received his Ph.D. degree in physical chemistry from Indiana University and M.S. degree in computer science from Rochester Institute of Technology. He has worked on industry research since 1979 in areas of inkjet ink testing with Mead Office System, ink formulation with Xerox, and color technology development with Xerox, Peerless, and Aetas. Currently, he is an independent consultant with expertise in areas of color technology (algorithm development and implementation), color system architecture (image path, performance evaluation, and modeling), digital halftoning, digital image processing, and image quality (evaluation, assessment, and improvement). He is the author of two books, an article in IEEE encyclopedia, and over 30 papers in areas of color device characterization and calibration, color mixing model, color image processing, and digital halftoning.