# Extending Kubelka-Munk's Theory with Lateral Light Scattering

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## Abstract

Due to its simplicity, the theory of KUBELKA-MUNK [1] has found a wide acceptance for modeling the optical properties of light scattering materials. However, the concept is not explicitly adapted to predict halftone prints on paper. In this respect, a recent improvement was given by BERG [2]. Our approach is an extension of BERG's model in order to reduce the gap between the mathematical description of the paper's point spread function and the experimental results of simple reflectance measurements.

## 1. Introduction

Lateral light scattering is often modeled by a point spread function (*PSF*). In the literature, several approaches of the point spread function for paper are known. Most of them were determined empirically, see for instance [3], or assume a specific type of function [4, 5] but others are based on numerical simulations [6, 7], microscopic reflectance measurements [8], image processing [9], multi-flux theory [10] or radiative diffusion [11].

The original theory of KUBELKA and MUNK [1] was developed for light diffusing and absorbing infinitely wide colorant layers. Due to its simple use and to its acceptable prediction accuracy, this model is very popular in industrial applications. The concept is based on the simplified picture of two diffuse light fluxes through the layer, one proceeding downward and the other simultaneously upward. Instead of applying the KM-model on a colorant layer, we adopt this theory for analyzing the light scattering inside paper printed with a halftone.

Halftone ink or toner prints consists of microscopically varying transmittances and cannot be regarded as an infinitely wide layer. Incident photons being scattered into the lateral direction can be re-scattered again in the original perpendicular direction after traveling a period along the extension of the paper. Consequently, the entrance point of incident light differs form its exit point in general which is well known as the YULE-NIELSON [3] effect. Because of the negligence of lateral light streams, the original concept of KUBELKA-MUNK is not adequate for modeling optical properties of halftone prints. To overcome this drawback, BERG [2] has introduced isotropic light scattering into a KM oriented approach. However, his model remains restricted to two dimensions and disregards the influences of brightened papers or surface reflections.

In this article we extend BERG's differential equation system to three dimensions. Especially, we analyze the spectral absorption and internal reflections within the ink or coating layers on both sides of a paper sheet. This is done by an appropriate enhancement of the boundary conditions of the differential equation system. Finally, we adapt the model to brightened papers by considering the fluorescence of the paper's bulk.

The solution is given by an analytical, fourier transformed description of the reflectance and transmittance factors as a function of different, optically relevant constants, in particular, the internal reflections, the transmittance variations of the printed ink layer and the specific scattering and absorbing factors.

## 2. Brightened papers

Basically, the point spread function of brightened papers is affected by the included fluorescent additives. The supplied brighteners absorb a certain energy of the radiations invisible upper frequency band, *the excitation spectrum* [12]. A specific amount of that energy, defined by the *quantum yield*, is then released by radiative relaxation in a visible band, *the fluorescence spectrum*. This technique compensates the yellowish appearance of natural non-brightened papers. But, as the emission of the fluorescence spectrum happens in all space directions in the paper's body, it acts like a diffuse partial light source or converter. By consequence, the brightening effect amplifies the point spread function of the paper in the bluish range of the visible light and must be taken into account when analyzing the PSF. This is done in the analysis of the following section.

## 3. Three Dimensional Scattering Analysis

First note that a circular illumination source together with a circular detection sensor results in circular symmetric measurements. For that reason, the efforts can be reduced by choosing polar coordinates for the problem formulation, see Fig. 1. Let $r$ and $z$ be the radial and the vertical coordinates, the azimuth is neglected due to symmetry.

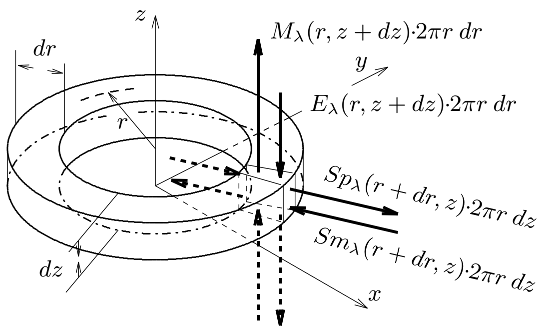


*Figure 1: Considered optical fluxes in a cylindrical paper section.*

Considering a cylindrical ring of a paper sheet of total thickness $D$, we apply a spectral analysis of the up- and downward oriented light fluxes ($M_\lambda$ and $E_\lambda$) as well as of the lateral ones ($Sp_\lambda$ and $Sm_\lambda$). Any light flux that enters the infinitesimal volume element $2\pi r\,\mathrm{d}r\,\mathrm{d}z$ from a specific direction, loses a fractional amount $\sigma\cdot 2\pi r\,\mathrm{d}r\,\mathrm{d}z$ by scattering and $\alpha\cdot 2\pi r\,\mathrm{d}r\,\mathrm{d}z$ by absorption and by brighteners excitation. At the same time, it gains an according amount by scattering and by fluorescing from the other fluxes. As it exits the volume element we can write for the case of $E_\lambda(r, z)$

$$\begin{aligned}2\pi r \cdot E_\lambda(r,z)\,\mathrm{d}r = {} & 2\pi r\cdot E_\lambda(r,z{+}\mathrm{d}z)\,\mathrm{d}r - \\ & 2\pi r\cdot(\sigma_{dp}{+}\sigma_{dm}{+}\sigma_{du}{+}\alpha)\cdot E_\lambda(r,z{+}\mathrm{d}z)\,\mathrm{d}r\,\mathrm{d}z + \\ & 2\pi r\cdot(\sigma_{pd}{+}\phi_f)\cdot Sp_\lambda(r,z)\,\mathrm{d}r\,\mathrm{d}z + \\ & 2\pi r\cdot(\sigma_{md}{+}\phi_f)\cdot Sm_\lambda(r{+}\mathrm{d}r,z)\,\mathrm{d}r\,\mathrm{d}z + \\ & 2\pi r\cdot(\sigma_{ud}{+}\phi_f)\cdot M_\lambda(r,z)\,\mathrm{d}r\,\mathrm{d}z\,,\end{aligned} \tag{1}$$

with the spectral paper parameters:

- $\alpha$ specific extinction coefficient of the paper and of the fluorescent additives according to [12, eq. 1],
- $\sigma_{1\,2}$ specific scattering coefficient for a light beam coming from direction 1 and being scattered to direction 2, *used direction acronyms:* ***d****own,* ***u****p,* ***m****inus,* ***p****lus*,
- $\phi_f$ normalized fluorescent spectrum, weighted by the quantum yield [12, eq. 2].

For the remaining fluxes $M_\lambda$, $Sp_\lambda$ and $Sm_\lambda$ similar balances can be derived according to Fig. 1 and Eq. (1). The factor $2\pi r$ can be canceled out of all of them. Note that in a strong mathematical sense, the introduced optical constants retain their validity only in the applied polar coordinates. A verifying comparison of these parameters to established ones from a cartesian system is therefore subject to an appropriate parameter transformation. From a more practical point of view, the physical relevance of these parameters is given by their correspondence with the empirical data.

Analog to KM and BERG, we assume a linear variation of the fluxes along $\mathrm{d}r$ and $\mathrm{d}z$ which allows us to ignore terms of higher order. Hence, applying a Taylor series expansion, we obtain a system of coupled linear partial differential equations

$$\begin{aligned}\frac{\partial E_\lambda(r,z)}{\partial z} = {} & (\sigma_{dp}{+}\sigma_{dm}{+}\sigma_{du}{+}\alpha)\cdot E_\lambda(r,z) - \\ & (\sigma_{pd}{+}\phi_f)\cdot Sp_\lambda(r,z) - (\sigma_{md}{+}\phi_f)\cdot Sm_\lambda(r,z) - \\ & (\sigma_{ud}{+}\phi_f)\cdot M_\lambda(r,z)\,, \\[1ex] \frac{\partial M_\lambda(r,z)}{\partial z} = {} & -(\sigma_{up}{+}\sigma_{um}{+}\sigma_{ud}{+}\alpha)\cdot M_\lambda(r,z) + \\ & (\sigma_{pu}{+}\phi_f)\cdot Sp_\lambda(r,z) + (\sigma_{mu}{+}\phi_f)\cdot Sm_\lambda(r,z) + \\ & (\sigma_{du}{+}\phi_f)\cdot E_\lambda(r,z)\,, \\[1ex] \frac{\partial Sp_\lambda(r,z)}{\partial r} = {} & -(\sigma_{pd}{+}\sigma_{pu}{+}\sigma_{pm}{+}\alpha)\cdot Sp_\lambda(r,z) + \\ & (\sigma_{dp}{+}\phi_f)\cdot E_\lambda(r,z) + (\sigma_{up}{+}\phi_f)\cdot M_\lambda(r,z) + \\ & (\sigma_{mp}{+}\phi_f)\cdot Sm_\lambda(r,z)\,, \\[1ex] \frac{\partial Sm_\lambda(r,z)}{\partial r} = {} & (\sigma_{md}{+}\sigma_{mu}{+}\sigma_{mp}{+}\alpha)\cdot Sm_\lambda(r,z) - \\ & (\sigma_{dm}{+}\phi_f)\cdot E_\lambda(r,z) - (\sigma_{um}{+}\phi_f)\cdot M_\lambda(r,z) - \\ & (\sigma_{pm}{+}\phi_f)\cdot Sp_\lambda(r,z)\,.\end{aligned} \tag{2}$$

## 4. Boundary Conditions

Choosing adequate boundary conditions is a crucial task. On one hand, they strongly affect the accuracy and the complexity of the model. On the other hand, they build the connecting link between pure mathematics and observed physics.

We assume a circularly illuminated printed paper sheet placed on a backing that has a spectral reflectance factor $R_{b_\lambda}$. Since the paper is a translucent media, the optical conditions at the paper's top and bottom boundaries must be considered. In the proposed concept, whereas both faces can be covered by a coated or printed ink layer[1] (see Fig. 2), only top-face *halftone* prints are allowed. At these interfaces, any light flux that passes through the piled layers are subject to multiple reflections [13]. To consider this, we adopt the SAUNDERSON correction [14] which determines the fraction of light being transmitted or reflected by the paper's interfaces[2]. At the bottom-face of the paper's bulk, we approximate the spectral fraction of light reflected from the bottom coating and backing interface by $\rho_{pb}$. In the case of transmittance measurements, the fraction of light transmitted through the backing plate and the bottom coating layer is approximated by $\tau_{bp}$

$$\rho_{pb} = (\rho_i + \tau_i \cdot \tau_s \cdot \frac{R_{b_\lambda}}{1 - \rho_s \cdot R_{b_\lambda}}) \cdot \mathcal{T}_{b_\lambda}^2 , \quad (3)$$

$$\tau_{bp} = \frac{\tau_s \cdot \mathcal{T}_{b_\lambda}}{1 - \rho_i \cdot \sigma_{ud} \cdot \mathcal{T}_{b_\lambda}^2} , \quad (4)$$

with the spectral parameters:

$\rho_s$, $\tau_s$ Fresnel reflectance and transmittance of the coating-air or ink-air interfaces,

$\rho_i$, $\tau_i$ corresponding internal Fresnel reflectance and transmittance factors,

$\mathcal{T}_{b_\lambda}$ internal transmittance of the bottom-face coating or inked layer.

Similarly, the top surface transmits into the paper a fraction approximated by $\tau_{ap}$ and a fraction out of the paper approximated by $\tau_{pa}$

$$\tau_{ap}(r) = \frac{\tau_s(r) \cdot \mathcal{T}_{t_\lambda}(r)}{1 - \rho_i(r) \cdot \sigma_{du} \cdot \mathcal{T}_{t_\lambda}(r)^2} , \quad (5)$$

$$\tau_{pa}(r) = \tau_i(r) \cdot \mathcal{T}_{t_\lambda}(r) , \quad (6)$$

with $\mathcal{T}_{t_\lambda}(r)$ the point transmittance of the top-face coating or inked layer which could be a halftone print. SAUNDERSON applied his fractions to correct the "reflectance $\mathcal{R}$, which the sample would have if measured in a transparent medium having the same index of refraction as the sample". Since we impose these corrections inherently on the boundary condition, the "reflectance $\mathcal{R}$" is replaced by the "reflecting" scattering coefficients $\sigma_{ud}$ and $\sigma_{du}$.

Further, in order to obtain an accurate fit of reflectance measurements, a "specular reflectance" $\rho_{ap}$ must be taken into account. According to our experience, best results are achieved by assuming:

$$\rho_{ap}(r) = \rho_s \cdot \mathcal{T}_{t_\lambda}(r)^2 . \quad (7)$$

[1] In this article, only non-fluorescent and weekly scattering inks are assumed.

[2] Generally, these optical layers are much thinner than the paper's body. Therefore, when analyzing the paper's PSF, we neglect the lateral spreading of the optical fluxes within these layers, which occurs due to multiple internal reflections.

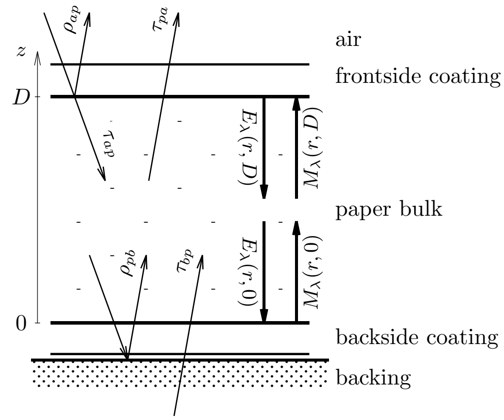


*Figure 2: Scheme of the considered boundary conditions.*

Applying these corrections, the boundary conditions for the reflectance factor of a paper sheet of thickness $D$ are given by:

$$M_\lambda(r,0) = \rho_{pb} * E_\lambda(r,0) , \quad (8)$$

$$E_\lambda(r,D) = \tau_{ap} \cdot E_{0_\lambda} , \quad (9)$$

where $*$ indicates a two dimensional convolution. Accordingly, the boundary conditions for the transmittance factor can be written as:

$$M_\lambda(r,0) = \rho_{pb} * E_\lambda(r,0) + \tau_{bp} \cdot M_{0_\lambda} , \quad (10)$$

$$E_\lambda(r,D) = 0 . \quad (11)$$

Finally, the model for the measured reflectance and transmittance factor can be obtained by solving the corresponding boundary condition problem and applying the appropriate corrections:

$$R_\lambda(r) = \rho_{ap}(r) + \tau_{pa}(r) * M_\lambda(r,z)\big|_{z=D} , \quad (12)$$

$$T_\lambda(r) = \tau_{pa}(r) * M_\lambda(r,z)\big|_{z=D} . \quad (13)$$

## 5. Paper Modulation Transfer Function

The type of the obtained partial differential equation system Eq. (2) is the same as in BERG's thesis. The system can be simplified by a one dimensional Fourier transform [15] along the radius $r$. This permits to cancel the third and fourth partial part of the transformed equation system. Unfortunately, we have observed a difference to BERG's simplifying trace [2, Eq. 9.13–16] which makes it difficult to compare the results directly.

$E_{0_\lambda}$ and $M_{0_\lambda}$ from Eq. (9 and 10) are set to 1. For solving the boundary condition problem Eq. (12 and 13), only a semi-isotropic case is considered where all distracting and all reflecting scattering coefficients are identical respectively:

$$\sigma_x = \sigma_s \quad \text{for } x \in \{dp, pd, dm, md, up, pu, um, mu\}\,,$$
$$\sigma_x = \sigma_r \quad \text{for } x \in \{ud, du, pm, mp\}\,,$$

For the semi-isotropic case we obtain the point reflectance factor

$$R_\lambda(r) = \rho_{ap}(r) + \tau_{ap}(r) \cdot \mathcal{F}^{-1}\Big[\mathcal{F}\big[\tau_{pa}(r)\big] \cdot \frac{(e^{2D\sqrt{\frac{c_{1_\rho}}{c_{2_\rho}}}}+1)\,\rho_{pb}\,c_{1_\rho} - (e^{2D\sqrt{\frac{c_{1_\rho}}{c_{2_\rho}}}}-1)\,(c_{3_\rho}+\rho_{pb}\,c_{4_\rho})}{(e^{2D\sqrt{\frac{c_{1_\rho}}{c_{2_\rho}}}}+1)\,c_{1_\rho} + (e^{2D\sqrt{\frac{c_{1_\rho}}{c_{2_\rho}}}}-1)\,(\rho_{pb}\,c_{3_\rho}+c_{4_\rho})}\Big]\,, \quad (14)$$

and the point transmittance factor

$$T_\lambda(r) = \tau_{bp} \cdot \mathcal{F}^{-1}\Big[\mathcal{F}\big[\tau_{pa}(r)\big] \cdot \frac{2\,c_{1_\rho}\,e^{D\sqrt{\frac{c_{1_\rho}}{c_{2_\rho}}}}}{(e^{2D\sqrt{\frac{c_{1_\rho}}{c_{2_\rho}}}}+1)\,c_{1_\rho} + (e^{2D\sqrt{\frac{c_{1_\rho}}{c_{2_\rho}}}}-1)\,(\rho_{pb}\,c_{3_\rho}+c_{4_\rho})}\Big]\,, \quad (15)$$

with:

$\rho$ spatial frequency,

$\mathcal{F}$ the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

and

$$c_{1_\rho} = (\alpha+2\,\sigma_r+2\,\sigma_s+\phi_f) \cdot \big[(\alpha+2\,\sigma_r+2\,\sigma_s+\phi_f)\,(\alpha-3\,\phi_f)\,(\alpha+4\,\sigma_s+\phi_f) + \rho^2\,(\alpha+2\,\sigma_s-\phi_f)\big]\,,$$

$$c_{2_\rho} = \rho^2 + (\alpha+2\,\sigma_s-\phi_f)\,(\alpha+2\,\sigma_r+2\,\sigma_s+\phi_f)\,,$$

$$c_{3_\rho} = -2 \cdot \Big[\rho^2\,(\sigma_r+\phi_f) + (\alpha+2\,\sigma_r+2\,\sigma_s+\phi_f) \cdot \big[2\,\sigma_s^2+\sigma_r\,(\alpha+2\,\sigma_s-\phi_f) + (\alpha+6\,\sigma_s)\,\phi_f+\phi_f^2\big]\Big]\,,$$

$$c_{4_\rho} = 2\,\rho^2\,(\alpha+\sigma_r+2\,\sigma_s) + 2\,(\alpha+2\,\sigma_r+2\,\sigma_s+\phi_f) \cdot \big[2\,\sigma_s^2+\sigma_s\,(4\,\alpha-6\,\phi_f)+\sigma_r\,(\alpha+2\sigma_s-\phi_f) + (\alpha-2\,\phi_f)(\alpha+\phi_f)\big]\,. \quad (16)$$

As printed color layers behave like spectral light filters, they can reduce the energy of the fluorescence spectrum of the paper by absorbing a part of the excitation spectrum of the fluorescent additives. In this case, $\phi_f$ must be multiplied by a factor $\delta\phi_f$ in order to diminish the fluorescent spectrum.

Until now, it was impossible to find a closed inverse transformation back into the original spatial domain. However, Eq. (14 and 15) are well suited for fast numerical transformation by an inverse *fft* routine. Actually, the kernel of these equations can be interpreted as a Modulation Transfer Function of paper.

## 6. Conclusion

For an accurate reflectance prediction of a halftone print, we propose an extended method which is based on the concepts of KUBELKA-MUNK and the thesis of BERG. By taking the dominant optical scattering properties of common papers into account, it is possible to fit the model for non-uniform color transmittances of the print. In the present article, we introduce a mathematical description of the paper's point spread function, which can be fitted to reflectance and transmittance measurements of simple solid patches.

In particular, we have estimated the numerical values of the introduced coefficients for different papers and succeeded to fit numerical results of spectral measurements with an accuracy of $1\%$ and less. This was performed by applying a numerical transformation of the model and optimizing the coefficients in order to fit measurements of the bare paper and of several prints of solid patches simultaneously.

## 7. Biography

Safer Mourad received the M.S. degree in electrical engineering in 1993 from the Swiss Federal Institute of Technology (ETH), Zurich. From 1993 to 1997, he developed algorithms for real-time video tracking and high-end surveying instruments at Leica Geosystems. In 1998 he joined the Media Technology Department at the EMPA, St. Gallen and is currently working towards his Ph. D. His research interests includes colorimetry, image processing and real-time control applications. He received the Swiss IEEE Award for his M.S. thesis and is a member of the IS&T.
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