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Reflection Density in Photographic Color Prints: Generalizations of the Williams–Clapper Transform

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We present two generalizations of the Williams–Clapper transform for converting between transmission and reflection densities of a color print. The first generalization allows for arbitrary incident angle, viewing angle, and index of refraction. The second generalization is to the geometry of an integrating sphere with the specular reflection either included or excluded. Our derivation also clarifies a potential source of confusion: Williams and Clapper had noted that, because of the cancellation of two factors, a reflection print with a perfectly transmitting gelatin layer (and a base of perfect reflectance) has the same brightness as it would have in the absence of the gelatin layer. We find that this cancellation is in fact only approximate. The approximate nature of the cancellation was not readily apparent because of the fortuitous closeness of the cancellation for the particular geometry and indices of refraction considered by Williams and Clapper. We also discuss efficient numerical implementation of the transform and outline an example of an application.

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Introduction

In 1953, Williams and Clapper published a seminal paper in which they derived the transformation necessary for converting between the transmission density of the gelatin layer and the resulting reflection density for a color print.[1,2] Their calculation takes into account the multiple internal reflections that can occur in the gelatin layer and are very important at low densities. They only presented equations for determining the reflection density for the specific case of a 45°/0° geometry (i.e., light is incident at an angle of 45° to the paper normal and is viewed along the normal direction); it was also assumed that the gelatin coating has an index of refraction of 1.53.

Since the publication of Williams and Clapper, published work related to transmission-to-reflection density transformations for color prints has proceeded along four lines. One has been the comparison of the Williams–Clapper result to actual experimental data,[3] which has generally shown it to work quite well. A second has been the introduction of various approximations to the Williams–Clapper formula that do not involve evaluation of the integral (which was computationally difficult and costly in the early days of computers).[3–5] The transformation has also sometimes been derived from fits to experimental data, using either purely empirical forms[6] or the above-mentioned approximate expressions with certain values in the expression taken as free parameters.[4]

A third line of research has been the modification of the Williams–Clapper transformation by adding an additional empirical term to the reflectance that controls the maximum reflection density.[3–8] This term is usually attributed to light reflected off the first-surface (gelatin–air interface) back to the viewer, although it, can also be due in part to light scattered back to the viewer from within the gelatin layer (e.g., off of the dye droplets).[9] Because the addition of such a term is straightforward and is dealt with in so many other presentations, we will not explicitly include it in the discussions below except for the case of an integrating sphere where its magnitude can be calculated theoretically.

A fourth line has been the generalization of Williams–Clapper to different geometries. The two generalizations of which we are aware are the unpublished work of Watt,[7] in which the ratio of the reflectance in an integrating sphere geometry (with an input angle of 8° to the normal and specular reflection excluded) to B/B' in the 45°/0° geometry was calculated, and the work of Ohta,[8] in which the reflection density was calculated for the case of diffuse (Lambertian) illumination with an arbitrary viewing angle.

Finally, we note that a formal mathematical approach to the problem of light absorption and scattering in color prints based on the Kubelka–Munk model[10] has very recently been presented by Emmel and Hersch.[11] It is noted there that an extension of the approach to the situation, where the light is no longer necessarily normal to the print,[12] yields the Williams–Clapper result as a special case.

In this study, we consider the transformation between transmission and reflection density for two qualitatively different geometries. One is for arbitrary incident and viewing angles; the other is for an arbitrary incident angle and an integrating sphere viewing geometry with

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specular reflection either included or excluded. In the course of the derivation, we endeavor to explain the origin of each term in the. transform in full detail. We also shed light on a potential source of confusion that arises because of a fortuitous cancellation of factors for the specific geometry and indices of refraction studied by Williams and Clapper. Following this, a brief discussion of efficient numerical implementation of the transforms is presented and some results are shown comparing transforms using the different geometries. Finally, we outline one potential application of our results.

The Transform for Arbitary Incident and Viewing Angles

The first geometry that we consider is the same as that in Ref. 1 except that the incident and viewing angles and the refractive indices can take on arbitrary values. We assume the base is a diffuse (Lambertian) reflector with reflectance R_B and that the gelatin has a transmittance t along a single base–to–interface path normal to the base. Note that we label quantities in the outside medium ("air") by the subscript 1, and in the inside medium ("gelatin") by the subscript 2; for simplicity, we will often suppress the subscript on angles that are integrated over. We also superscript angles by 'i' for incident angle and "v" for viewing angle. The ratio of the brightness B of the color print to the brightness B' of a perfect diffuse white reflector identically illuminated in air is given by

$$\frac{B}{B'} = \left(1 - r_{\theta_2^i}\right)\left(1 - r_{\theta_2^v}\right) R_B t^{\sec\left(\theta_2^i\right)+\sec\left(\theta_2^v\right)} \frac{\cos\left(\theta_2^v\right) d\Omega_2^v}{\cos\left(\theta_1^v\right) d\Omega_1^v} \sum_{j=0}^{\infty} \left[\frac{\int R_B t^{2\sec(\theta)} r_\theta \cos(\theta) d\Omega}{\int \cos(\theta) d\Omega}\right]^j, \quad (1)$$

where the definitions of the various symbols appearing here and the physical meaning of each of the terms are explained in detail in the paragraphs that follow.

The first term on the right-hand side of Eq. 1 represents the fraction of the incident light that is transmitted into the gelatin; the second term represents that fraction of the returning light (on a path through the gelatin that would take it to the viewer) that is transmitted into the air, and R_B represents the fraction of light reflected at the base. Here r_{θ_2} is the fresnel reflectance of unpolarized light.[13] Following Ref. 1 and others, we use the convention that the angle subscripting r is always that in the gelatin medium and also suppress explicitly writing the dependence on n_2/n_1. The expression for the fresnel reflectance is

$$r_{\theta_2} = \frac{1}{2}\left|\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}\right|^2 + \frac{1}{2}\left|\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}\right|^2, \quad (2)$$

where

$$\sin(\theta_1) = \frac{n_2}{n_1}\sin(\theta_2) \quad (3)$$

via Snell's Law.[13,14] By including the absolute value signs in (2), we correctly get $r_{\theta_2} = 1$ for the case of total internal reflection, i.e., when $(n_2/n_1)\sin(\theta_2) > 1$ and θ_1 is thus complex. The symmetry of (2) with respect to θ_1 and θ_2 makes it clear that the expression for fresnel reflectance is independent of which side of the interface the light is incident upon. (That is, the fresnel reflectance for light, incident on the interface from medium 1 at an angle θ_1 is equal to that for light incident from medium 2 at an angle θ_2, where θ_1 and θ_2 are related through Snell's law.) This is what has allowed us to write the fraction of the. incident light that enters the gelatin as simply $(1 - r_{\theta_i})$.

The term in Eq. 1 involving the transmittance t accounts for the light absorbed upon passing through the gelatin layer. The exponent on t is the total path length of the light into and out of the material in units where the perpendicular path length used to define the transmittance t is one. This does not include the path length due to internal reflections at the gelatin–air interface that is accounted for in the final term, as discussed below.

Perhaps the subtlest factor in Eq. 1 is the ratio

$$\frac{\cos(\theta_2^v) d\Omega_2^v}{\cos(\theta_1^v) d\Omega_1^v}.$$

This is a conversion factor between the brightness B_2 in the gelatin and the brightness B_1 in the air. It is necessary because the refraction of the light at the interface causes the light through a small solid angle

$$\delta\Omega_2^v\left[\equiv \sin\left(\theta_2^v\right)\delta\theta_2^v\delta\phi_2^v\right]$$

in the gelatin to spread into a solid angle $\delta\Omega_1^v$ upon going from the gelatin to the air. It is derived by requiring that the energy passing through these solid angles be equal in the gelatin and the air (neglecting the reflected portion):[15]

$$B_1 \cos(\theta_1^v)\delta\Omega_1^v = B_2 \cos(\theta_2^v)\delta\Omega_2^v. \quad (4)$$

Thus,

$$\frac{B_1}{B_2} = \frac{\cos(\theta_2^v) d\Omega_2^v}{\cos(\theta_1^v) d\Omega_1^v}, \quad (5)$$

which is the desired result. By taking Snell's law (Eq. 3), differentiating both sides, and then multiplying each side by Snell's law once again, one can show that

$$\frac{\cos(\theta_2^v) d\Omega_2^v}{\cos(\theta_1^v) d\Omega_1^v} = \left(\frac{n_1}{n_2}\right)^2. \quad (6)$$

Finally, the last term in Eq. 1 accounts for light that is internally-reflected, perhaps multiple times. With each additional internal reflection, there is an additional term from reflections at the interface and at the base and a term from the absorption during the additional path length of travel of $2\sec(\theta)$ through the gelatin. The integrals are over the solid angle in the upper half-plane, with $d\Omega \equiv \sin(\theta)d\theta d\phi$. As in Eq. 4, the geometrical projection factor of $\cos(\theta)$ accounts for the distribution of light from a diffuse reflector.[15]

Upon substituting Eq. 6 into Eq. 1, summing the geometric series, and using the normalization

$$2\int_0^{\pi/2} \cos(\theta)\sin(\theta)d\theta = 1, \quad (7)$$

we arrive at our final result:

$$\frac{B}{B'} = \frac{\left(1 - r_{\theta_2^i}\right)\left(1 - r_{\theta_2^v}\right) t^{\sec\left(\theta_2^i\right) + \sec\left(\theta_2^v\right)} \left(n_1 / n_2\right)^2}{\dfrac{1}{R_B} - I(t)}, \qquad (8)$$

where

$$I(t) \equiv 2\int_0^{\pi/2} t^{2\sec(\theta)} r_\theta \cos(\theta)\sin(\theta)d\theta. \qquad (9)$$

A notable feature of Eq. 8 is that the result is unchanged under the interchange of incident and viewing angles, as expected on the basis of the Helmholtz Reciprocity Principle.[16] One can verify that Eq. 8 reproduces the equation given by Williams and Clapper for the special case of $n_2/n_1 = 1.53$, $\theta_1^i = 45°$, and $\theta_1^v = 0°$.

In their study, Williams and Clapper find that the reflection density $[\equiv -\log_{10}(B/B')]$ is 0 for $R_B = 1$ and $t = 1$. They note that this is due to a cancellation of two factors, namely the geometric series for the multiple reflections produces a factor of 2.59 which is then cancelled by a factor of 1/2.59 due to the reflections and refraction at the gelatin–air interface, i.e., $(0.945)(0.956)/1.53^2$. It seems tempting to believe that this is a consequence of energy conservation and holds generally. However, the truth is that this is not the case. The cancellation is not exact, and the extreme closeness of the cancellation is a somewhat fortuitous consequence of the particular geometry and index of refraction considered. Although one can make vague arguments as to why such cancellation might occur approximately, and indeed the cancellation is within a few percent over a fairly wide range of angles and indices of refraction, as demonstrated in Fig. 1.

What is true from energy conservation for $R_B = 1$ and $t = 1$ is not that the brightnesses at each angle be equal, but rather that the *total* energy emitted at all angles from the print must equal the total energy incident on the print. This can be restated to say that the total energy reflected from the print must be equal to that reflected from a perfect diffuse reflector in air. To check energy conservation for Eq. 8 we note

$$\begin{aligned}\frac{\text{Energy out}}{\text{Energy in}} &= r_{\theta_2^i} + \frac{\int B\cos(\theta_1^v)d\Omega_1^v}{\int B'\cos(\theta_1^v)d\Omega_1^v} \\ &= r_{\theta_2^i} + 2\left(\frac{n_2}{n_1}\right)^2 \int_0^{\pi/2} \frac{B}{B'}\cos(\theta_2^v)\sin(\theta_2^v)d\theta_2^v. \end{aligned} \qquad (10)$$

The first equality follows from the fundamental relation between brightness and photometric illumination,[15] and also includes the additional term for the light that is specularly reflected off of the first-surface. In equating the second and third expressions, we have used Eqs. 6 and 7, and the fact that B' is independent of angle (by the definition of a diffuse reflector). Upon substituting Eq. 8 with $R_B = 1$ and $t = 1$ into Eq. 10 and simplifying, we confirm that (Energy out)/(Energy in) = 1.

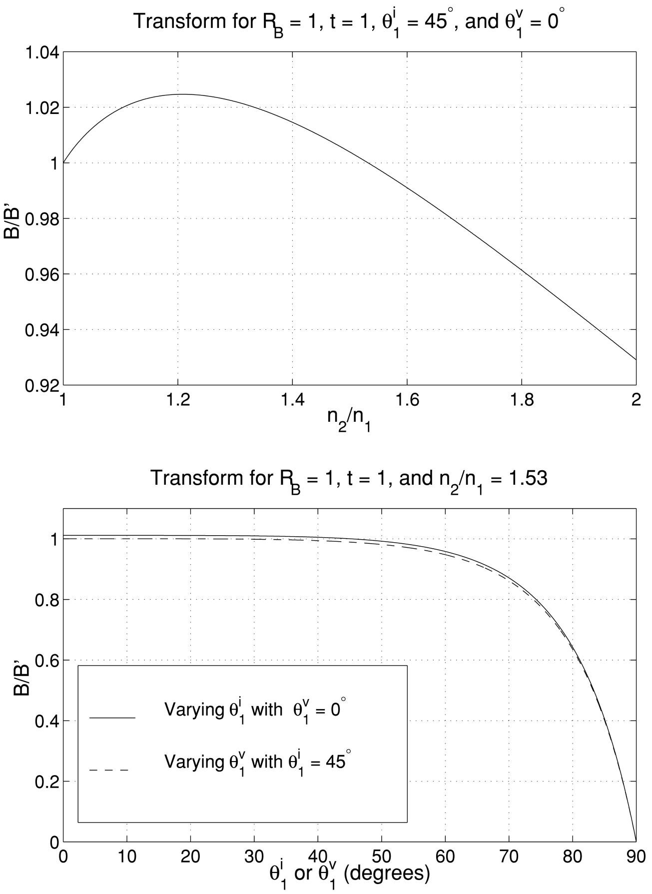

Figure 1. Generalized Williams–Clapper transform showing the ratio of the brightness B for a color print with $R_B = 1$ and $t = 1$ to the brightness B' of a diffuse white reflector identically illuminated in air. The top plot shows how B/B' varies as a function of n_2/n_1 for $\theta_1^i = 45°$ and $\theta_1^v = 0°$. The bottom plot shows how B/B' varies as a function of the incident angle for the viewing angle fixed at $\theta_1^v = 0°$ (solid curve) and as a function of θ_1^v for fixed $\theta_1^i = 45°$ (dashed curve). In both cases, the ratio of refractive indices is $n_2/n_1 = 1.53$. Note that there is a rather large range of n_2/n_1, θ_1^i, and θ_1^v for which B/B' is within a few percent of 1. However, the *extreme* closeness of B/B' to 1 for those particular values studied by Williams and Clapper ($\theta_1^i = 45°$, $\theta_1^v = 0°$, and $n_2/n_1 = 1.53$) is clearly seen to be fortuitous.

The Transform for an Integrating Sphere Geometry

We now discuss the generalization of the Williams–Clapper result to handle an integrating sphere viewing geometry. As noted above, this case was treated by Watt,[7] although in computing the ratio of the reflectance for this case to B/B' for the Williams–Clapper (45°/0°) geometry, he assumed that both would be precisely equal to 1 when $R_B = 1$ and $t = 1$. As noted above, this assumption is almost but not exactly true for the 45°/0° geometry and, as noted below, is even less for the integrating sphere geometry with the specular reflection excluded. By computing only this ratio, he also did not present an explicit expression for the integrating sphere geometry.

For this geometry, the argument proceeds exactly as for the derivation of Eq. 8 except that in the end it is necessary to integrate over all viewing angles in a half-sphere. Because of this, the relevant quantity is no longer the ratio of brightnesses B/B' but is now a true reflectance R, i.e., the total fraction of the incident light that is reflected back. (B/B' *is* sometimes itself referred to as a reflectance.[3,7,8] This is a slight abuse of terminology which we avoid here.) We already outlined this integration step above when checking energy conservation and the result thus follows immediately from substituting Eq. 8 into Eq. 10. For an integrating sphere which includes the specularly-reflected light off the first surface, the result is

$$R = r_{\theta_2^i} + \frac{(1-r_{\theta_2^i})\, t^{\sec(\theta_2^i)} I^*(t)}{\dfrac{1}{R_B} - I(t)}, \qquad (11)$$

when $I(t)$ is given in Eq. 9 and

$$I^*(t) \equiv 2\int_0^{\pi/2} t^{\sec(\theta)} (1 - r_\theta) \cos(\theta) \sin(\theta) d\theta. \qquad (12)$$

If the integrating sphere excludes the specularly reflected light, then the first term on the righthand side of Eq. 11 does not appear. For the case where the specular light is included, we have already noted above that $R = 1$ when $R_B = 1$ and $t = 1$, as required by energy conservation. Note that in the excluded case, the reflectance for $n_2/n_1 = 1.53$ is reduced from 1 by at least 4%.

In retrospect, because the equations for the behavior of light are symmetric under interchange of the direction of travel of the light, it can be argued that this integrating sphere result should be equal to the result for *B/B'* found by Ohta,[8] for the case of a color print under diffuse illumination viewed at a particular angle, provided that one substitutes the incident angle here for the viewing angle in that case. Comparing Eq. 11 [minus the specular term] with Eq. 29 of Ref. 8 does show them to be the same with the recognition that the definition of the integrals here and those there are related by

$$I_1 = \frac{1}{2} I(t)$$

and

$$I_2 = \frac{1}{2} \left(\frac{n_2}{n_1} \right)^2 I^*(t);$$

the factor of $(n_2/n_1)^2$ appears because Ohta performs the integration with respect to the angle outside the gelatin medium.

Numerical Implementation of the Transforms and Some Results

We have implemented the calculation of the above generalizations of the Williams–Clapper transform in MATLAB. The calculation of the functions $I(t)$ and $I^*(t)$ that contain integrals is straightforward in MATLAB using built-in numerical quadrature routines. However, if the transformation must be performed repeatedly for different values of t, R_B, θ_1^i or θ_1^v, these computations can become quite expensive. This problem is resolved as follows. For a given value of the ratio of indices of refraction, n_2/n_1, the functions $I(t)$ and $I^*(t)$ are calculated only once at a few hundred values of t. The results are then fit by a rational quadratic spline. This makes subsequent calculations of the transform for various values of R_B, t, θ_1^i and θ_1^v computationally inexpensive.

Figure 2 shows a systematic comparison of the transmission density to reflection density transform for the 45°/0° geometry and the 0° integrating sphere geometry with the specular part either excluded or included. Note that the transforms for the 45°/0° geometry and the 0° integrating sphere geometry with the specular part excluded are very similar and this probably explains why those two particular geometries are often used virtually interchangeably. By contrast, the transform for the integrating sphere geometry with the specular part included shows a saturation of the reflection density due to the specular reflection off of the gelatin–air interface.

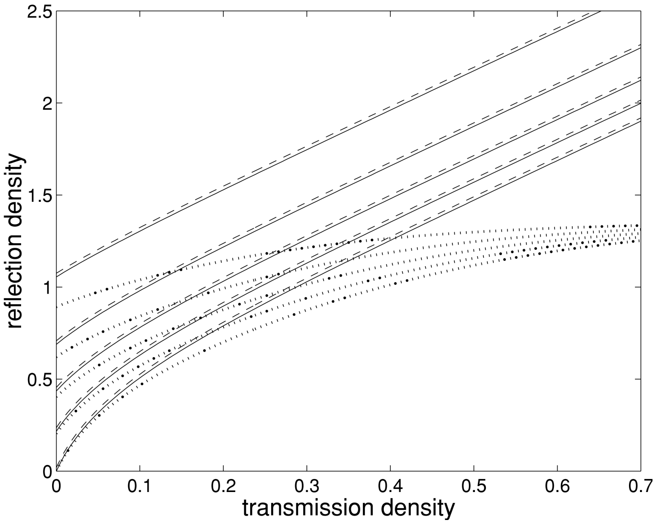

Figure 2. A comparison of the transmission density to reflection density transform for the 45° incident 0° viewing geometry and the 0° integrating sphere geometry with the specular part either excluded or included. The reflection density is shown as a function of transmission density for R_B = 0.2, 0.4, 0.6, 0.8 and 1.0 (top to bottom). The solid lines are for the 45°/0° geometry, the dashed lines are for the integrating sphere geometry with the specular part excluded, and the dotted lines are for the integrating sphere geometry with the specular part included. In all cases, the ratio of refractive indices is $n_2/n_1 = 1.53$.

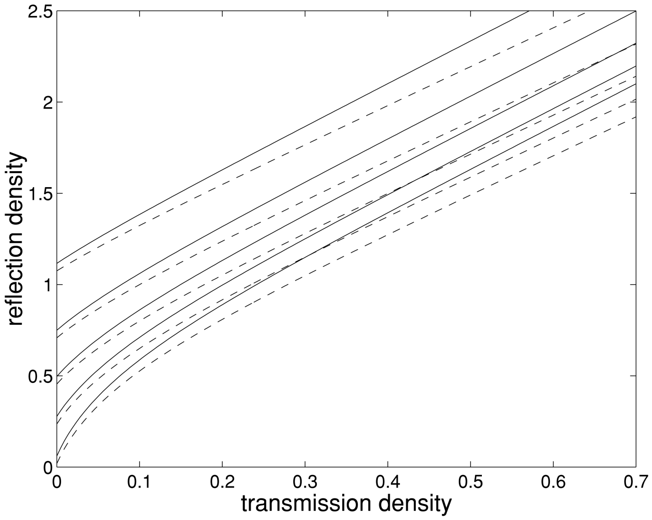

Figure 3. Same as Fig. 2, except now the solid lines show the transform for a 70° incident/30° viewing geometry. (For clarity, the transform for the integrating sphere geometry with the specular part included is not shown).

Figure 3 compares the transform for a 70°/30° and a 0° integrating sphere geometry with the specular part excluded. Because the path length through the gelatin for the 70°/30° geometry is longer than the average path length for the 0° integrating sphere geometry, this plot of reflection density as a function of transmission density has a steeper slope at the high densities for the former case than for the latter.

A Potential Application

The main focus of this study is on the derivation of the transmission density to reflection density transformation rather than its application. However, for illustrative purposes we will now briefly discuss one possible application.

Consider a system consisting of several elements: a light source, a print containing (for the purposes of this illustration) a single dye, a filter, and a detector. Each of these elements has certain spectral characteristics. For example, the light source has a certain spectral intensity $I(\lambda)$, the reflectance of the paper with the dye is given by $R(\lambda)$, the transmittance of the filter is $T(\lambda)$, and the quantum efficiency of the detector is $D(\lambda)$; here λ is the wavelength of the light. The total response of the detector is then simply proportional to the integral of the product of the spectral response functions of each of the elements:

$$\text{Response} \sim \int I(\lambda)R(\lambda)T(\lambda)D(\lambda)d\,\lambda. \tag{13}$$

The desire to optimize this system in the sense of determining the spectral characteristics of, say, the filters and the dye that give the largest possible “contrast ratio” (change in the response) between areas of the print where the dye is present in a certain quantity and areas of the print where the the dye is absent.

The above problem is fairly straightforward to solve; however, the one complicating factor is the reflectance of the print. Because this reflectance might be measured in one standard geometry but desired for a detector set up in various other geometries, it is necessary to employ our generalizations of the Williams–Clapper transform to convert between transmittance and reflectance. The use of the transform proceeds as follows: First, we perform the transform in reverse in order to determine the transmittance of the paper containing the dye, given data on the reflectance in the measured geometry. The transform is then performed in the forward direction for the geometry of interest in order to determine the reflectance in that new geometry.

Note that the use of the transmission density to reflection density transform in the simple form that we have presented involves various assumptions, e.g., that the dye is uniform across a wide enough area that the spatial non-uniformity need not be explicitly considered in the transform, that scattering of light is unimportant, and that the paper backing is a Lambertian diffuse reflector. All these assumptions are expected to hold to a good approximation; in any event, to the extent that the reflectance in the measured and desired geometries will likely be fairly similar, any errors introduced are likely to be rather small.

Conclusions

We have discussed two generalizations of the Williams–Clapper transform for converting between transmission and reflection densities in a color print. The first generalization allows for arbitrary incident angle, viewing angle, and index of refraction. The second generalization is to the geometry of an integrating sphere with the specular reflection either included or excluded.

We have also pointed out that a cancellation of two factors that Williams and Clapper noted occurs, and leads to a print with the same brightness as in the absence of the gelatin layer in the case of a coating with perfect transmittance and a base with perfect reflectance, is only approximate. In particular, the cancellation is not required by energy conservation; the approximate nature of this cancellation is not readily apparent because of the fortuitous closeness of the cancellation for the particular geometry and indices of refraction considered by Williams and Clapper.

Finally, we have discussed efficient numerical implementation of the transform, have shown some comparisons of the transform for different geometries, and have outlined one potential application.

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14. Both the numerator and denominator of Eq. 2 are zero for 0_2 Æ 0. However, by taking the limit 0_2 in Eq. 2, one finds
15. M. Born and E. Wolf (Ref. 13), See. 4.8.
16. See F. J. J, Clarke and D. J. Parry, *Lighting Res. and Tech*. **17,** 1 (1985). In practice, polarization effects could lead to some violation of reciprocity. It holds in our equation because, following Ref. 1, we have assumed that the illuminating light is unpolarized and, furthermore, have made the approximation of ignoring any polarization of the light that occurs due to the various reflections and refractions.