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Superfast joint demosaicing and super-resolution

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Abstract

The problem of increasing spatial resolution from Bayer images is considered. It is solved locally at each point of the unknown high-resolution image. In each local neighbourhood, subpixel warp and blur kernel are assumed to be the same, which makes it possible to reduce the computational complexity from $\mathscr{O}\left(n^6\right)$ to $\mathscr{O}\left(n^2 \log n\right)$. A detailed description of the algorithm and its proof using the apparatus of multilevel matrices are provided. The relation between solutions of the SR problem with different warping parameters is also studied, and it is proven that certain solutions can be derived from solutions with other parameters by using simple transforms. This makes it possible to reduce the amount of memory used for storing filters within a filter bank approach up to 80 times (2× magnification, 3 input frames).

Introduction

Multi-frame super-resolution (SR) is reconstruction of a high-resolution image X from several observed low-resolution images Y_i. Solving this problem starts with an agreement on an image formation model like [1]:

$$Y_i = W_i X + \eta_i, \forall i = 1, ..., k,$$

where W_i is the i^{th} image formation operator and η_i is additive noise. If $\eta_i = \eta$ is Gaussian white noise, the SR problem can be formulated as

$$X = \underset{X}{\operatorname{argmin}} || \sum_{i=1}^{k} (W_i X - Y_i) ||^2.$$

Operator W_i is an image formation operator, describing how each low-resolution was obtained from a high-resolution image. Input data is often insufficient for unique reconstruction. Therefore, a regularized problem is considered instead:

$$X = \underset{X}{\operatorname{argmin}} || \sum_{i=1}^{k} (W_i X - Y_i) ||^2 + \Gamma(X). \quad (1)$$

Different types of norms, regularization terms, and corresponding solvers are described in detail in [8]. Most of the research in SR is focused on the problem with a quadratic data fidelity term and total variation (TV) regularization term. In the case of a non-linear and particularly non-convex form of the regularization term in (1), the only way to find a solution is an iterative approach.

This paper considers a simpler $L_2 - L_2$ problem with $\Gamma(X) = \Lambda^2 (HX)^* (HX)$, where H is a convolution operator. This makes it possible to solve (1) using the linear equation

$$\hat{A} X = W^* Y, \quad (2)$$

where $\hat{A} = W^* W + \Lambda^2 H^* H, W = [W_1^*, \ldots, W_k^*]^*, Y = [Y_1^*, \ldots, Y_k^*]^*$. Operators W_i can be composed out of warp M_i, blur G_i, and decimation D for a single-channel SR problem, as in [3], [2], and [4]:

$$W_i = D G M_i. \quad (3)$$

Alternatively, Bayer decimation B may be used for reconstruction from a Bayer domain (joint demosaicing and super-resolution) as in [8] and [6]:

$$W_i = B D G M_i. \quad (4)$$

It is possible to make this problem even narrower and assume each warp M_i and blur G as being space invariant. This limitation is quite reasonable when processing a small image patch([2]). In this case, $\hat{A}$, meeting condition (3), is known to be reducible to a block diagonal form ([3], [2], [4], [7]), while the case of (4) has never been covered in literature. The main contribution of this paper is obtaining a block diagonal form for $\hat{A}$ meeting (4). The section titled *Matrix classes arising from the SR problem* discusses the transformation of 1D, 2D, and Bayer SR problems to a multilevel class with diagonal block classes on different levels. The section *Block diagonalization for SR* derives the explicit form of transforms to obtain the block diagonal form from matrix classes discussed in the previous section. In the section *Symmetries in the Bayer SR problem* filter bank implementation from [6] is recollected, and symmetry properties that allow a reduction in the amount of memory used to store the filter bank are studied. The sub-sections titled *Prerequisites* in each section summarize basic facts used in the proofs.

Matrix classes arising from the SR problem

Prerequisites

This paper will extensively use a permutation matrix P obtained from the identity matrix by row permutation. Left multiplication of the matrix M by P causes permutation of rows, while right multiplication by $Q = P^T$ leads to the same permutation of columns. Permutation matrices P and Q are orthogonal:

$$P^{-1} = P^T, Q^{-1} = Q^T. \quad (5)$$

Notation P^u will mean cyclic shift by u, providing

$$(P^u)^* = (P^u)^{-1} = P^{-u}. \quad (6)$$

A perfect shuffle matrix $\Pi_{n_1 n_2}$(notes on application to structured matrices can be found in [10]) will also be needed. It corresponds to the transposition of a rectangular matrix of size $n_1 \times n_2$ in vectorized form. This is an $n \times n$ permutation matrix where $n = n_1 n_2$ and an element with indices i, j is one if and only if i and j can be presented as $i - 1 = \alpha_2 n_1 + \alpha_1, j - 1 = \alpha_1 n_2 + \alpha_2$ for some integers $\alpha_1, \alpha_2 : 0 \leq \alpha_1 \leq n_1 - 1, 0 \leq \alpha_2 \leq n_2 - 1$.

An explicit formula for the matrix of a Fourier transform of size $n \times n$ will also be used:

$$F_n = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & \varepsilon_n^{1\cdot 1} & \dots & \varepsilon_n^{1\cdot(n-2)} & \varepsilon_n^{1\cdot(n-1)} \\ 1 & \varepsilon_n^{2\cdot 1} & \dots & \varepsilon_n^{2\cdot(n-2)} & \varepsilon_n^{2\cdot(n-1)} \\ & & \vdots & & \\ 1 & \varepsilon_n^{(n-1)\cdot 1} & \dots & \varepsilon_n^{(n-1)\cdot(n-2)} & \varepsilon_n^{(n-1)\cdot(n-1)} \end{bmatrix}, \quad (7)$$

where $\varepsilon_n = e^{-\frac{2\Pi i}{n}}$. The Fourier matrix and its conjugate satisfy

$$F_n^* \cdot F_n = F_n \cdot F_n^* = n \cdot I_n. \quad (8)$$

Definition 1. A circulant matrix is a matrix with a special structure, where every row is a right cyclic shift of the row above

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ & & \ddots & \\ a_2 & a_3 & \dots & a_1 \end{bmatrix}.$$

and corresponds to 1D convolution with cyclic boundary conditions. Circulant matrices are invariant under cyclic permutations

$$\forall A \in \mathbb{C} \Rightarrow A = (P^u)^T A P^u. \quad (9)$$

The class of circulant matrices of size $n \times n$ is denoted by $\mathbb{C}_n$, so it is possible to write $A \in \mathbb{C}_n$. A circulant matrix is defined by a single row (or column) $a = [a_1, a_2, \dots, a_n]$. It can be transformed to diagonal form by Fourier transform:

$$\forall A \in \mathbb{C}_n \Rightarrow A = \frac{1}{n} F_n^* \Lambda_n F_n. \quad (10)$$

From (10), it follows directly that all circulant matrices of the same size commute. Many matrices used below are circulant e.g., matrices corresponding to 1D convolution with cyclic boundary conditions.

As far as this paper is going to deal with 2 or more dimensions, the Kronecker product $\otimes$ becomes an important tool. An operator that down-samples a vector of length n by the factor s can be written as $D_s = I_{n/s} \otimes e_{1,s}^T$, where $e_{1,s}^T$ is the first row of identity matrix I_s. Suppose a 2-dimensional $n \times n$ array is given:

$$X^{matr} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ & & \vdots & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix}.$$

In vectorized form, this can be written as

$$X^T = [x_{11}, x_{21}, \dots, x_{n1}, x_{12}, x_{22}, \dots, x_{n2}, \dots, x_{1n}, \dots, x_{nn}].$$

If A_n is a 1D convolution operator from $\mathbf{R}^n$ to $\mathbf{R}^n$ with coefficients $a_i, i = 1, ..., n$, then $I_n \otimes A_n$ applied to X will correspond to row-wise convolution acting from $\mathbf{R}^{n^2}$ to $\mathbf{R}^{n^2}$, and $A_n \otimes I_n$ – to column-wise convolution with this filter. For two row-wise and column-wise convolution operators A_n and B_n operator $A_n \otimes B_n$ will be a separable convolution operator from $\mathbf{R}^{n^2}$ to $\mathbf{R}^{n^2}$ for vectorized $n \times n$ arrays due to the following property of the Kronecker product:

$$AB \otimes CD = (A \otimes C)(B \otimes D). \quad (11)$$

For example, 2D down-sampling by factor s will be

$$D_{s,s} = D_s \otimes D_s = I_{n/s} \otimes e_{1,s}^T \otimes I_{n/s} \otimes e_{1,s}^T. \quad (12)$$

Two-dimensional non-separable convolution (warp and blur) operators are block circulant with circulant block (BCCB) and can be expressed via a sum of Kronecker products of 1D convolution operators:

$$\forall A \in \mathbb{C}_n \mathbb{C}_m \Rightarrow \exists N_i \in \mathbb{C}_n, M_i \in \mathbb{C}_m, i = 1..r : A = \sum_{i=1}^{r} N_i \otimes M_i. \quad (13)$$

From (10), (11) and (13) follows easily an explicit formula of transformation of a BCCB matrix to block diagonal form:

$$\forall A \in \mathbb{C}_n \mathbb{C}_m \Rightarrow A = \frac{1}{mn} (F_n^* \otimes F_m^*) \Lambda (F_n \otimes F_m), \quad (14)$$

where $\Lambda = \sum_{i=1}^{r} \Lambda_i^N \otimes \Lambda_i^M$ and Λ_i^N, Λ_i^M are diagonal matrices of eigenvalues of matrices N_i and M_i from (13).

Although BCCB matrices and their properties are extensively covered in literature, matrices arising from the SR problem (especially the Bayer case) are more complicated, and this paper will borrow a more general concept of matrix class from [5] to deal with them in a simple and unified manner.

Definition 2. A matrix class is a linear subspace of square matrices. Matrix A with elements $a_{i,j} : i, j = 1..n$ belongs to matrix class $\mathbb{M}$ described by numbers $a_{ij}^{(q)}, q \in Q$ if it satisfies

$$\sum_{i,j} a_{ij}^{(q)} a_{ij} = 0. \quad (15)$$

This paper will consider classes of square matrices $\mathbb{C}_n$(circulant), $\mathbb{G}$ (general, $Q = \varnothing$) and $\mathbb{D}_n$ (diagonal) of size $n \times n$.

If matrix blocks satisfy some conditions of form (15), it is possible to say that this matrix belongs to the particular block class. The Kronecker product produces bilevel matrices of class $\mathbb{M}^1\mathbb{M}^2$ from matrices from classes $\mathbb{M}^1$ and $\mathbb{M}^2$: $\forall M^1 \in \mathbb{M}^1, \forall M^2 \in \mathbb{M}^2 \Rightarrow M^1 \otimes M^2 \in \mathbb{M}^1\mathbb{M}^2$. Here, $\mathbb{M}^1$ is called an outer class and $\mathbb{M}^2$ an inner class. Saying $A \in \mathbb{G}\mathbb{M}$ simply means that each block of A belongs to class $\mathbb{M}$. Multi-level classes like $\mathbb{M}^1\mathbb{M}^2 \dots \mathbb{M}^i \dots \mathbb{M}_j \dots \mathbb{M}^k$ can be also constructed. This section classifies matrices from different SR problems and transforms them to some multi-level classes $\mathbb{M}^1\mathbb{M}^2 \dots \mathbb{M}^i \mathbb{D} \mathbb{M}^{i+1} \dots \mathbb{M}^j \mathbb{D} \mathbb{M}^{j+1} \dots \mathbb{M}^k$ containing several diagonal block classes, where $\mathbb{M}^i$ stand for some non-diagonal types.

1D case

The single-channel SR matrix $\hat{A}$ from problem (2) uses formation model (3) and can be expanded as

$$\hat{A} = \left(\sum_{i=1}^{k} M_i^* G^* D^* D G M_i \right) + \Lambda^2 H^* H. \quad (16)$$

In 1D M_i and H being convolution operators provides

$$M_i, G, H \in \mathbb{C}_n, D = D_s = I_{n/s} \otimes e_{1,s}^T. \quad (17)$$

Theorem 1. *Matrix* (16) *meeting conditions* (17) *satisfies*

$$\hat{A} = F_n^* \Lambda_A F_n,$$

where $\Lambda_A \in \mathbb{G}_s\mathbb{D}_{n/s}$.

To prove theorem 1, the following is necessary:

Lemma 1. Let $\mathbf{1}_{s,s}$ be an $s \times s$ matrix of all ones. Then

$$F_n D_s^* D_s F_n^* = (n/s)\, \mathbf{1}_{s,s} \otimes I_{n/s}.$$

Proof. First,

$$D_s F_n^* = \mathbf{1}_{1,s} \otimes F_{n/s}^* \tag{18}$$

will be proved. According to (7), an element of matrix F_n^* with indices m,k is $F_n^*[m,k] = \varepsilon_n^{-(m-1)(k-1)}$. Elements of the matrix $U = D_s F_n^*$ of size $n \times ns$ will be $F_n^*[1+s(l-1),k], l = 1,\ldots,n/s, k = 1,\ldots,n$; i.e., $U[l,k] = \varepsilon_n^{-s(l-1)(k-1)}$. Notice that

$$\varepsilon_n^{-s(l-1)(k-1)} = \varepsilon_{n/s}^{-(l-1)((k-1) \bmod (n/s))}. \tag{19}$$

From (19) follows that $U = [\underbrace{F_{n/s}^*, \ldots, F_{n/s}^*}_{s \text{ times}}]$, which coincides with (18). From (8), (11) and (18):

$$F_n D_s^* D_s F_n^* = \left(\mathbf{1}_{s,1} \otimes F_{n/s}\right)\left(\mathbf{1}_{1,s} \otimes F_{n/s}^*\right) = \tfrac{n}{s} \mathbf{1}_{s,s} \otimes I_{n/s}.$$

□

Now it is possible to prove theorem 1.

Proof. Property (10) implies

$$\hat{A} = \left(\textstyle\sum_{i=1}^k \left(\tfrac{1}{n} F^* \Lambda_i F\right)^* \left(\tfrac{1}{n} F^* \Lambda_G F\right)^* D^* D \left(\tfrac{1}{n} F^* \Lambda_G F\right)\left(\tfrac{1}{n} F^* \Lambda_i F\right)\right) + \Lambda^2 \left(\tfrac{1}{n} F^* \Lambda_H F\right)^* \left(\tfrac{1}{n} F^* \Lambda_H F\right).$$

From (8) follows

$$\hat{A} = \tfrac{1}{n^2} F^* \left(\left(\textstyle\sum_{i=1}^k \Lambda_i^* \Lambda_G^* F D^* D F^* \Lambda_i \Lambda_G\right) + \Lambda^2 n \Lambda_H^* \Lambda_H\right) F. \tag{20}$$

Applying Lemma 1 to (20), $\hat{A} = \frac{1}{ns} F^* \Lambda_A F$ is obtained, where $\Lambda_A = \left(\sum_{i=1}^k \Lambda_i^* \Lambda_G^* \left(\mathbf{1}_{s,s} \otimes I_{n/s}\right) \Lambda_i \Lambda_G\right) + \Lambda^2 s \Lambda_H^* \Lambda_H$. As far as $\mathbf{1}_{s,s} \otimes I_{n/s} \in \mathbb{G}_s\mathbb{D}_{n/s}$, multiplication rules for block matrices yield $\Lambda_i^* \Lambda_G^* \left(\mathbf{1}_{s,s} \otimes I_{n/s}\right) \Lambda_i \Lambda_G \in \mathbb{G}_s\mathbb{D}_{n/s}$. Obviously from $\Lambda_H^* \Lambda_H \in \mathbb{D}_n$ immediately follows $\Lambda_H^* \Lambda_H \in \mathbb{G}_s\mathbb{D}_{n/s}$ and $\Lambda_A \in \mathbb{G}_s\mathbb{D}_{n/s}$. □

2D case

In the 2D case, warp, blur, and regularization operators become

$$M_i, G, H \in \mathbb{C}_n\mathbb{C}_n, D = D_{s,s} = D_s \otimes D_s. \tag{21}$$

Theorem 2. *Matrix* (16)*, meeting condition* (21)*, satisfies*

$$\hat{A} = (F_n^* \otimes F_n^*) \Lambda_A (F_n \otimes F_n),$$

where $\Lambda_A \in \mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s}$.

Proof. Equations (8) and (14) make it possible to rewrite (16) satisfying (21) as

$$\hat{A} = (1/n)^4 (F^* \otimes F^*)\Big(\big(\sum_{i=1}^k \Lambda_i^* \Lambda_G^* (F \otimes F) D_{s,s}^* D_{s,s} (F^* \otimes F^*) \Lambda_G \Lambda_i\big) + \Lambda^2 n^2 \Lambda_H^* \Lambda_H\Big)(F \otimes F). \tag{22}$$

From lemma 1, (12), and (11) immediately follows

$$(F \otimes F) D_{s,s}^* D_{s,s} (F^* \otimes F^*) = \left(\frac{n}{s}\right)^2 \mathbf{1}_{s,s} \otimes I_{n/s} \otimes \mathbf{1}_{s,s} \otimes I_{n/s}.$$

Hence, (22) simplifies to $\hat{A} = \frac{1}{n^2 s^2} (F_n^* \otimes F_n^*) \Lambda_A (F_n \otimes F_n)$, where $\Lambda_A = \left(\sum_{i=1}^k \Lambda_i^* \Lambda_G^* \left(\mathbf{1}_{s,s} \otimes I_{n/s} \otimes \mathbf{1}_{s,s} \otimes I_{n/s}\right) \Lambda_G \Lambda_i\right) + \Lambda^2 s^2 \Lambda_H^* \Lambda_H$. As far as both $\mathbf{1}_{s,s} \otimes I_{n/s} \otimes \mathbf{1}_{s,s} \otimes I_{n/s} \in \mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s}$ and $\Lambda_H^* \Lambda_H \in \mathbb{D} \in \mathbb{D}_n\mathbb{D}_n \in \mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s}$, the theorem is proven. □

Bayer case

In the Bayer case, vector X can be represented as stacked vectorized G,B, and R channels. Then matrix $\hat{A}$ from problem (2) uses formation model (4) and can be expanded as

$$\hat{A} = \left(\sum_{i=1}^k \tilde{M}_i^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i\right) + \Lambda^2 \tilde{H}^* \tilde{H}, \tag{23}$$

where

$$\tilde{D} = I_3 \otimes D_{s,s}, \tilde{G} = I_3 \otimes G, \tilde{M}_i = I_3 \otimes M_i,$$
$$B = \begin{bmatrix} D_{2,2} & 0 & 0 \\ D_{2,2}P^{1,1} & 0 & 0 \\ 0 & D_{2,2}P^{1,0} & 0 \\ 0 & 0 & D_{2,2}P^{0,1} \end{bmatrix},$$
$$\tilde{H} = \begin{bmatrix} H_g & 0 & 0 \\ 0 & H_b & 0 \\ 0 & 0 & H_r \\ H_{c_1} & -H_{c_1} & 0 \\ H_{c_2} & 0 & -H_{c_2} \\ 0 & H_{c_3} & -H_{c_3} \end{bmatrix}, \tag{24}$$

and $P^{u,v}$ is a 2D cyclic shift by u columns and v rows. Regularization operator $\tilde{H}$ is constructed so that it has both an inter-channel term (which may be different for each channel) and a cross-channel term (which may also be selected independently for each pair of channels). Sub-matrices from (24) satisfy

$$M_i, G, H_r, H_g, H_b, H_{c_1}, H_{c_2}, H_{c_3} \in \mathbb{C}_n\mathbb{C}_n. \tag{25}$$

Bayer down-sampling operator B from (24) extracts and stacks channels G_1, G_2, R and B from the pattern in Fig. 1.

Figure 1. *Bayer pattern*

Theorem 3. *Matrix* (23)*, meeting conditions* (24) *and* (25)*, satisfies*

$$\hat{A} = (I_3 \otimes F_n^* \otimes F_n^*)\Lambda_A(I_3 \otimes F_n \otimes F_n),$$

where $\Lambda_A \in \mathbb{G}_3\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}$.

Proof. Applying the multiplication rule for block matrices and (5) obtains

$$B^*B = \begin{bmatrix} D_{2,2}^*D_{2,2} + (P^{1,1})^*D_{2,2}^*D_{2,2}P^{1,1} & 0 & 0 \\ 0 & (P^{1,0})^*D_{2,2}^*D_{2,2}P^{1,0} & 0 \\ 0 & 0 & (P^{0,1})^*D_{2,2}^*D_{2,2}P^{0,1} \end{bmatrix}.$$

If $\tilde{M}_i^*\tilde{G}^*\tilde{D}^*B^*B\tilde{D}\tilde{G}\tilde{M}_i$ is expanded, non-diagonal blocks will be zero, and diagonal blocks will contain the term $D_{s,s}^*(P^{v,u})^*D_{2,2}^*D_{2,2}P^{v,u}D_{s,s}$ multiplied from the left and right sides by some other matrices. Taking into account that $D_{2,2}D_{s,s} = D_{2s,2s}$ and the fact that shifting by (v,u) after down-sampling s times is equivalent to shifting by (sv,su) before down-sampling, the following is obtained:

$$D_{s,s}^*(P^{v,u})^*D_{2,2}^*D_{2,2}P^{v,u}D_{s,s} = (P^{sv,su})^*D_{2s,2s}^*D_{2s,2s}P^{sv,su}. \quad (26)$$

So it is possible to rewrite

$$\tilde{D}^*B^*B\tilde{D} = \begin{bmatrix} \hat{D} + (P^{s,s})^*\hat{D}P^{s,s} & 0 & 0 \\ 0 & (P^{s,0})^*\hat{D}P^{s,0} & 0 \\ 0 & 0 & (P^{0,s})^*\hat{D}P^{0,s} \end{bmatrix}, \quad (27)$$

where $\hat{D} = D_{2s,2s}^*D_{2s,2s}$. As long as $P^{u,v} \in \mathbb{C}_n\mathbb{C}_n$, it commutes with blur and warp. Hence, it is possible to introduce supplementary variables $M_i' = P^{s,0}M_i, M_i'' = P^{0,s}M_i$, and $M_i''' = P^{s,s}M_i$ and use (26) to rewrite the expression under the sum in (23) as

$$\tilde{M}_i^*\tilde{G}^*\tilde{D}^*B^*B\tilde{D}\tilde{G}\tilde{M}_i = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix},$$

$$a_1 = M_i^*G^*D_{2s,2s}^*D_{2s,2s}GM_i + M_i'''^*G^*D_{2s,2s}^*D_{2s,2s}GM_i''',$$
$$a_2 = M_i'^*G^*D_{2s,2s}^*D_{2s,2s}GM_i',$$
$$a_3 = M_i''^*G^*D_{2s,2s}^*D_{2s,2s}GM_i''.$$

Theorem 2 provides that for $k = 1,\ldots,3$

$$(F_n \otimes F_n)a_k(F_n^* \otimes F_n^*) \in \mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}},$$

which means

$$(I_3 \otimes F_n \otimes F_n)\tilde{M}_i^*\tilde{G}^*\tilde{D}^*B^*B\tilde{D}\tilde{G}\tilde{M}_i^*(I_3 \otimes F_n^* \otimes F_n^*) \in$$
$$\in \mathbb{D}_3\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}} \subset \mathbb{G}_3\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}.$$

Each of the 3×3 blocks $h_{j,k}$ of matrix $\tilde{H}^*\tilde{H}$ satisfy

$$(F_n \otimes F_n)h_{j,k}(F_n^* \otimes F_n^*) \in$$
$$\in \mathbb{D} \subset \mathbb{D}_n\mathbb{D}_n \subset \mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s} \subset \mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}},$$

providing $(I_3 \otimes F_n \otimes F_n)\tilde{H}^*\tilde{H}(I_3 \otimes F_n^* \otimes F_n^*) \in \mathbb{G}_3\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}$ which finally gives

$$(I_3 \otimes F_n \otimes F_n)\hat{A}(I_3 \otimes F_n^* \otimes F_n^*) \in \mathbb{G}_3\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}.$$

□

Block diagonalization for SR

In papers relying on block diagonalization of BCCB matrices like [4], it is usually only noted that certain matrices can be transformed to $\mathbb{DG}$, and no explicit transforms are provided.

This section derives closed-form permutations rearranging matrices from classes $\mathbb{G}_s\mathbb{D}_{n/s}$, $\mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s}$ and $\mathbb{G}_3\mathbb{G}_{2s}\mathbb{D}_{n/(2s)}\mathbb{G}_{2s}\mathbb{D}_{n/(2s)}$ to block diagonal form $\mathbb{D}_{n/s}\mathbb{G}_s$, $\mathbb{D}_{n^2/s^2}\mathbb{G}_{s^2}$, and $\mathbb{D}_{n^2/(4s^2)}\mathbb{G}_{12s^2}$, respectively.

Prerequisites

The apparatus of multi-level matrices is very handy to grind this problem. Unfortunately, the most relevant explanation in English that could be found [9] does not contain statements exactly meeting the present needs, while necessary statements with proofs are available only in publication [5] (in Russian). As far as definition 2 is more narrow than in original work (the constant in the right-hand side of (15) has been set to zero, and only square matrices are used, while [5] studies arbitrary constants and rectangular matrices), simplified versions of appropriate statements are explained below.

Theorem 4. *Let* $\mathbb{M}^1$ *and* $\mathbb{M}^2$ *be two classes of* $n\times n$ *and* $m\times m$ *matrices, respectively. Then* $\forall A \in \mathbb{M}^1\mathbb{M}^2 : \Pi_{n,m}^T A \Pi_{n,m} \in \mathbb{M}^2\mathbb{M}^1$, *where* $\Pi_{n,m}$ *is a perfect shuffle.*

Theorem 4 is provided without proof, but its meaning is quite obvious from Fig. 2. A perfect shuffle matrix is used in the property of the Kronecker product [10]

$$\forall A \in \mathbf{R}^{n\times n}, \forall B \in \mathbf{R}^{m\times m} : (B \otimes A) = \Pi_{n,m}^T(A \otimes B)\Pi_{n,m}, \quad (28)$$

but theorem 4 is slightly more general. It can be applied to any class including $\mathbb{G}_n\mathbb{M}_m$ or $\mathbb{M}_n\mathbb{G}_m$, which is not necessarily expressible as $A \otimes B$ with $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{m\times m}$. The property of outer class preservation can be postulated for any matrix classes $\mathbb{M}^1$ of size $n\times n$ and $\mathbb{M}^2, \mathbb{M}^3$ of size $m\times m$ and for matrices P_m, Q_m providing $\forall A \in \mathbb{M}^2 \Rightarrow P_m^T A Q_m \in \mathbb{M}^3$:

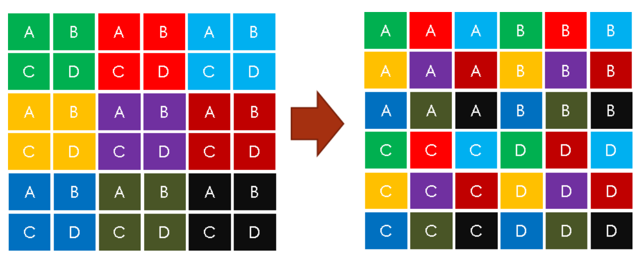

Figure 2. *Swapping matrix classes can be done by corresponding permutation of rows and columns*

$$\forall A \in \mathbb{M}^1\mathbb{M}^2 : (I_n \otimes P_m^T)A(I_n \otimes Q_m) \in \mathbb{M}^1\mathbb{M}^3. \quad (29)$$

This means that each inner block is transformed from class $\mathbb{M}^2$ to class $\mathbb{M}^3$, while the outer class $\mathbb{M}^1$, describing the interrelation between blocks, remains the same. From (29) and (28) immediately follows the property of inner class preservation. For matrix classes $\mathbb{M}^1, \mathbb{M}^3$ of size $n\times n$ and $\mathbb{M}^2$ of size $m\times m$, and matrices P_n, Q_n satisfying $\forall A \in \mathbb{M}^1 \Rightarrow P_n^T A Q_n \in \mathbb{M}^3$:

$$\forall A \in \mathbb{M}^1\mathbb{M}^2 : (P_n^T \otimes I_m)A(Q_n \otimes I_m) \in \mathbb{M}^3\mathbb{M}^2. \quad (30)$$

1D case

Theorem 5. *Matrix* (16)*, meeting condition* (17)*, satisfies*

$$\Pi^T_{s,\frac{n}{s}} F_n \hat{A} F_n^* \Pi_{s,\frac{n}{s}} \in \mathbb{D}_{n/s}\mathbb{G}_s.$$

Proof. The formula immediately follows from theorems 1 and 4. □

Theorem 5 provides that the 1D SR problem of dimension $n \times n$ can be reduced to $\frac{n}{s}$ sub-problems of dimension $s \times s$.

2D case

Theorem 6. *Matrix* (16)*, meeting condition* (21)*, satisfies*

$$(I_{\frac{n}{s}} \otimes \Pi^T_{s,\frac{n}{s}} \otimes I_s)\Pi^T_{ns,\frac{n}{s}}(F_n\otimes F_n)\hat{A}(F_n^*\otimes F_n^*)\Pi_{ns,\frac{n}{s}}(I_{\frac{n}{s}}\otimes \Pi_{s,\frac{n}{s}}\otimes I_s) \in \mathbb{D}_{\frac{n^2}{s^2}}\mathbb{G}_{s^2}.$$

Proof. From theorem 2 follows $\Lambda_A = (F_n \otimes F_n)\hat{A}(F_n^* \otimes F_n^*) \in \mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s}$. Theorem 4 provides $\Pi^T_{ns,\frac{n}{s}}\Lambda_A\Pi_{ns,\frac{n}{s}} \in \mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s$ and

$$\forall M \in \mathbb{G}_s\mathbb{D}_{n/s} \Rightarrow \Pi^T_{s,\frac{n}{s}} M \Pi_{s,\frac{n}{s}} \in \mathbb{D}_{n/s}\mathbb{G}_s. \quad (31)$$

Equations (29), (30) and (31) yield $\forall M \in \mathbb{D}_{n/s}\mathbb{G}_s\mathbb{D}_{n/s}\mathbb{G}_s \Rightarrow (I_{\frac{n}{s}} \otimes \Pi^T_{s,\frac{n}{s}} \otimes I_s)M(I_{\frac{n}{s}} \otimes \Pi_{s,\frac{n}{s}} \otimes I_s) \in \mathbb{D}_{\frac{n}{s}}\mathbb{D}_{\frac{n}{s}}\mathbb{G}_s\mathbb{G}_s$. Finally, $\mathbb{D}_{\frac{n}{s}}\mathbb{D}_{\frac{n}{s}}\mathbb{G}_s\mathbb{G}_s \subset \mathbb{D}_{\frac{n^2}{s^2}}\mathbb{G}_{s^2}$. □

Bayer case

Theorem 7. *Matrix* (23)*, meeting conditions* (24) *and* (25)*, satisfies*

$$\Pi^T_{3,n^2}\left(I_3 \otimes \left((I_{\frac{n}{2s}} \otimes \Pi^T_{2s,\frac{n}{2s}} \otimes I_{2s})\Pi^T_{2ns,\frac{n}{2s}}\right)\right)(I_3 \otimes F_n \otimes F_n)\hat{A}\cdot$$
$$\cdot (I_3 \otimes F_n^* \otimes F_n^*)\left(I_3 \otimes \left(\Pi_{2ns,\frac{n}{2s}}(I_{\frac{n}{2s}} \otimes \Pi_{2s,\frac{n}{2s}} \otimes I_{2s}\right)\right)\Pi_{3,n^2}$$
$$\in \mathbb{D}_{\frac{n^2}{4s^2}}\mathbb{G}_{12s^2}.$$

Proof. From theorem 3 follows $(I_3 \otimes F_n \otimes F_n)\hat{A}(I_3 \otimes F_n^* \otimes F_n^*) \in \mathbb{G}_3\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}$. Theorem 6 and property (29) give

$$\forall M \in \mathbb{G}_3\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}}\mathbb{G}_{2s}\mathbb{D}_{\frac{n}{2s}} \Rightarrow$$
$$\left(I_3 \otimes \left((I_{\frac{n}{2s}} \otimes \Pi^T_{2s,\frac{n}{2s}} \otimes I_{2s})\Pi^T_{2ns,\frac{n}{2s}}\right)\right) M \left(I_3 \otimes \left(\Pi_{2ns,\frac{n}{2s}}(I_{\frac{n}{2s}} \otimes \Pi_{2s,\frac{n}{2s}} \otimes I_{2s}\right)\right) \quad (32)$$
$$\in \mathbb{G}_3\mathbb{D}_{\frac{n^2}{4s^2}}\mathbb{G}_{4s^2}.$$

After applying theorem 4 to (32) and taking into account $\mathbb{D}_{\frac{n^2}{4s^2}}\mathbb{G}_{4s^2}\mathbb{G}_3 \equiv \mathbb{D}_{\frac{n^2}{4s^2}}\mathbb{G}_{12s^2}$, the proof is completed. □

Symmetries in the Bayer SR problem

Motivation

This section explains the reason to study the symmetries of the SR problem. The closed-form solution of (2) can be written as $X = \hat{A}^{-1}W^*Y$. In [6], the redundant structure of $\tilde{A} = \hat{A}^{-1}W^*$ was exploited, and elements of the pre-computed matrix were extracted in the form of filters (Fig. 3). It was assumed that the accuracy of the motion estimation algorithm is quite limited, so only c quantization levels of sub-pixel motion can be considered. In the single-channel 2D SR problem with k input frames, $c^{2(k-1)}$ sets with ks^2 filters in each set are required (the first frame is always supposed to have zero motion). In the Bayer SR problem $(2c)^{2(k-1)}$ sets with $3k(2s)^2$ filters in each set will be needed. Even for the small values $c = 4, k = 3, s = 4$, filter bank tends to occupy quite a lot of memory (2.3Gb for 16×16 filters stored with double precision). To reduce this value, a scheme with filter bank compression (Fig. 4) is applied, where only a limited number of filters is actually stored, and the rest are obtained by using simple transforms of these stored values, derived from symmetry properties. To obtain the desired symmetry properties, additional restrictions will be imposed.

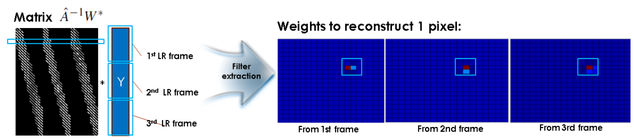

Figure 3. *Representing $\hat{A}^{-1}W^*$ in filter form*

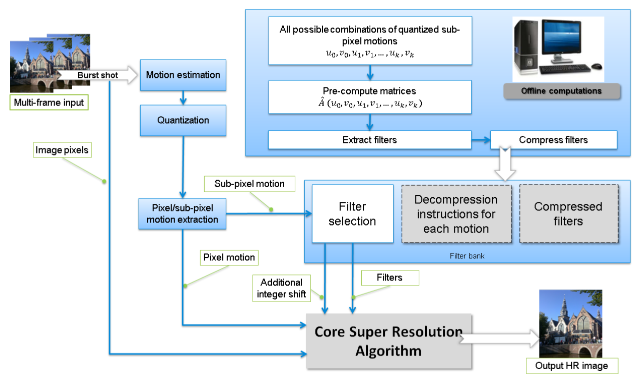

Figure 4. *Bayer pattern*

Additional restrictions

Conditions on the Bayer SR problem (equations (23), (24), and (25)) will be tightened and the following will be assumed:

$$H_r = H_g = H_b = \frac{1}{\sqrt{\gamma}}H_{c1} = \frac{1}{\sqrt{\gamma}}H_{c2} = \frac{1}{\sqrt{\gamma}}H_{c3} = H. \quad (33)$$

This condition is quite reasonable, because it assumes, that different image channels behave the same way. Condition (33) leads to

$$\tilde{H}^*\tilde{H} = \begin{bmatrix} 1+2\gamma & -\gamma & -\gamma \\ -\gamma & 1+2\gamma & -\gamma \\ -\gamma & -\gamma & 1+2\gamma \end{bmatrix} \otimes (H^*H).$$

$\mathbb{J}_3$ will denote a class of matrices sized 3×3 satisfying condition $\forall B \in \mathbb{J}_3 \implies JBJ^T = B$, where $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Obviously

$$J^*J = I. \quad (34)$$

Matrix $\tilde{H}^*\tilde{H}$, meeting (33), has a remarkable property

$$\tilde{H}^*\tilde{H} \in \mathbb{J}_3\mathbb{C}_n\mathbb{C}_n. \quad (35)$$

Warping will be considered to be integer cyclic shift in a high-resolution grid (which means $c = s$):

$$M_i = P^{u_i,v_i}, i = 1,...,k. \tag{36}$$

Consider a permutation matrix of size $n \times n$ that flips input vector

$$U_n = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & \dots & 0 & 0 \end{bmatrix},$$

which satisfies

$$U_n = U_n^* = U_n^{-1} \tag{37}$$

and

$$U_n P^u U_n = P^{-u}. \tag{38}$$

G and H are desired to satisfy the following conditions:

$$(U_n \otimes I_n)\, G\, (U_n \otimes I_n) = G, (U_n \otimes I_n)\, H\, (U_n \otimes I_n) = H, \tag{39}$$

$$(I_n \otimes U_n)\, G\, (I_n \otimes U_n) = G, (I_n \otimes U_n)\, H\, (I_n \otimes U_n) = H, \tag{40}$$

$$\Pi_{n,n}^T G \Pi_{n,n} = G, \Pi_{n,n}^T H \Pi_{n,n} = H. \tag{41}$$

In order to satisfy conditions (39), (40), and (41), G and H can be taken as convolution operators with some symmetric filters **H** and **G**, respectively. In the numeric experiments, **G** was chosen as being the Gaussian filter, and $\mathbf{H} = \begin{bmatrix} -1/8 & -1/8 & -1/8 \\ -1/8 & 1 & -1/8 \\ -1/8 & -1/8 & -1/8 \end{bmatrix}$.

Symmetric properties

Matrix $\hat{A}$ can be treated as a function of horizontal and vertical shifts describing M_i's: $\hat{A} = \hat{A}(u_1, v_1, u_2, v_2, \cdots, u_k, v_k)$.

The theorem below explains how to express solution for R values via solution for B values for the GBRG Bayer pattern.

Theorem 8. *Consider the transform* $\phi_1(\hat{A}) = (J \otimes P^{s,s})^* \hat{A}\, (J \otimes P^{s,s})$. *If conditions* (36) *and* (35) *are met then* $\hat{A}$ *from* (23) *satisfies*

$$\hat{A}(u_1, v_1, \cdots, u_k, v_k) = \phi_1(\hat{A}(u_1, v_1, \cdots, u_k, v_k)).$$

Proof. From (27) and (34) follows $\tilde{D}^* B^* B \tilde{D} = \phi_1(\tilde{D}^* B^* B \tilde{D})$, while from (24) and (34) follows $\tilde{G} = \phi_1(\tilde{G}), \tilde{M}_i = \phi_1(\tilde{M}_i)$. Equation (6) provides $(J \otimes P^{s,s})(J \otimes P^{s,s})^* = I$, yielding $\tilde{M}_i^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i = \phi_1(\tilde{M}_i^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i)$. From (35) immediately follows $\tilde{H}^* \tilde{H} = \phi_1(\tilde{H}^* \tilde{H})$. If $\tilde{H}^* \tilde{H} \in \mathbb{J}_3 \mathbb{C}_n \mathbb{C}_n$ then also $\tilde{H}^* \tilde{H} = \phi_1(\tilde{H}^* \tilde{H})$. □

Obviously, matrix $\tilde{H}^* \tilde{H}$, satisfying (33), also meets condition (35) of the theorem.

Theorem 9. *Consider the transform* $\phi_2(\hat{A}) = (I_3 \otimes P^{x,y})^* \hat{A}\, (I_3 \otimes P^{x,y})$ *and two integer numbers x and y. If condition* (35) *is met, then*

$$\hat{A}(u_1 + x, v_1 + y, \cdots, u_k + x, v_k + y) = \phi_2(\hat{A}(u_1, v_1, \cdots, u_k, v_k)).$$

Proof. Denote $\tilde{M}_i^{orig} = \tilde{M}_i(u_i, v_i)$ and

$$\tilde{M}_i^{shift} = \tilde{M}_i(u_i + x, v_i + y).$$

From (36) follows $P^{x,y} M_i = M_i P^{x,y}$. Consequently, $\tilde{M}_i^{orig}\,(I_3 \otimes P^{x,y}) = \tilde{M}_i^{shift}$, providing

$$\phi_2((\tilde{M}_i^{orig})^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i^{orig}) = (\tilde{M}_i^{shift})^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i^{shift}.$$

From (9) and (35), it also follows that $\tilde{H}^* \tilde{H} = \phi_2(\tilde{H}^* \tilde{H})$. □

Theorem 10. *Consider the transform* $\phi_3(\hat{A}) = (I_3 \otimes U_n \otimes I_n)\, \hat{A}\, (I_3 \otimes U_n \otimes I_n)$. *If condition* (39) *is met, then changing the sign of a single* u_i *for all input frames provides*

$$\hat{A}(-u_1 - 1, v_1, \cdots, -u_k - 1, v_k) = \phi_3(\hat{A}(u_1, v_1, \cdots, u_k, v_k)).$$

Proof. Denote

$$\tilde{M}_i^{urev} = \tilde{M}_i(-u_i - 1, v_i).$$

From (36) and (38) follows

$$(I_3 \otimes P^{-1,0})\,(I_3 \otimes U_n \otimes I_n)\,\tilde{M}_i^{orig}\,(I_3 \otimes U_n \otimes I_n) = \tilde{M}_i^{urev}.$$

Properties (38),(39), and (37) lead to

$$\tilde{G} \tilde{M}_i^{urev} = (I_3 \otimes P^{-1,0})(I_3 \otimes U_n \otimes I_n) \tilde{G} \tilde{M}_i^{orig} (I_3 \otimes U_n \otimes I_n).$$

The rest of the proof will be based on the fact that $U_n P^1 D_s^* D_s P^{-1} U_n = D_s^* D_s$, which makes it possible to rewrite

$$(I_3 \otimes U_n \otimes I_n)(I_3 \otimes P^{1,0}) \tilde{D}^* B^* B \tilde{D} (I_3 \otimes P^{-1,0})(I_3 \otimes U_n \otimes I_n) = \\ = \tilde{D}^* B^* B \tilde{D},$$

providing

$$(\tilde{M}_i^{urev})^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i^{urev} = \phi_3((\tilde{M}_i^{orig})^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i^{orig})$$

From (39) and (37) immediately follows $\tilde{H}^* \tilde{H} = \phi_3(\tilde{H}^* \tilde{H})$. □

This proof can be easily extended to

Theorem 11. *Consider the transform* $\phi_4(\hat{A}) = (I_3 \otimes I_n \otimes U_n)\, \hat{A}\, (I_3 \otimes I_n \otimes U_n)$. *From condition* (40) *follows*

$$\hat{A}(u_1, -v_1 - 1, \cdots, u_k, -v_k - 1) = \phi_4(\hat{A}(u_1, v_1, \cdots, u_k, v_k)).$$

Theorem 12. *Consider the transform*

$$\phi_5(\hat{A}) = (I_3 \otimes \Pi_{n,n}^T)^* \hat{A}\, (I_3 \otimes \Pi_{n,n}).$$

If condition (41) *is met, then*

$$\hat{A}(v_1 + s, u_1 + s, \cdots, v_k + s, u_k + s) = \phi_5(\hat{A}(u_1, v_1, \cdots, u_k, v_k)).$$

Proof. Denote $\tilde{M}_i^{swap} = \tilde{M}_i(v_i + s, u_i + s)$. From (28) immediately follows $P^{v,u} = \Pi_{n,n}^T P^{u,v} \Pi_{n,n}$, so taking into account (36) obtains $(I_3 \otimes P^{s,s})^* (I_3 \otimes \Pi_{n,n}^T)^* \tilde{M}_i^{orig}\, (I_3 \otimes \Pi_{n,n})\, (I_3 \otimes P^{s,s}) = \tilde{M}_i^{swap}$. Restriction (41) yields $(I_3 \otimes \Pi_{n,n}^T)^* \tilde{G}\, (I_3 \otimes \Pi_{n,n}) = \tilde{G}$.

Notice that $\Pi_{n,n}^T = \Pi_{n,n}, \Pi_{n,n} \Pi_{n,n} = I_{n^2}$ and $D_{s,s}^* D_{s,s} = \Pi_{n,n}^T D_{s,s}^* D_{s,s} \Pi_{n,n}$. Substituting these formulae into (27) yields

$$(I_3 \otimes \Pi_{n,n}^T) \tilde{D}^* B^* B \tilde{D} (I_3 \otimes \Pi_{n,n}) = (I_3 \otimes P^{s,s})^* \tilde{D}^* B^* B \tilde{D} (I_3 \otimes P^{s,s})$$

which provides

$$\phi_5((\tilde{M}_i^{orig})^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i^{orig}) = (\tilde{M}_i^{swap})^* \tilde{G}^* \tilde{D}^* B^* B \tilde{D} \tilde{G} \tilde{M}_i^{swap}.$$

Equation $\tilde{H}^* \tilde{H} = \phi_5(\tilde{H}^* \tilde{H})$ immediately follows from (41). □

Theorem 13. *If $\sigma(i)$ is any permutation of indices $i = 1,...,k$, then*

$$\hat{A}(u_1, v_1, \cdots, u_k, v_k) = \hat{A}(u_{\sigma(1)}, v_{\sigma(1)}, \cdots, u_{\sigma(k)}, v_{\sigma(k)}).$$

Proof. It immediately follows from (23). □

Theorem 14. *Adding $2s$ to one of the u_i's or v_i's does not change the problem:*

$$\hat{A}(u_1, v_1, \cdots, u_k, v_k) = \hat{A}(u_1, v_1, \cdots, u_{i-1}, v_{i-1}, u_i + 2s, v_i, u_{i+1}, v_{i+1} \cdots, u_k, v_k),$$
$$\hat{A}(u_1, v_1, \cdots, u_k, v_k) = \hat{A}(u_1, v_1, \cdots, u_{i-1}, v_{i-1}, u_i, v_i + 2s, u_{i+1}, v_{i+1} \cdots, u_k, v_k).$$

Proof. From (27) and $(P^{2s})^* D^*_{2s} D_{2s} P^{2s} = D^*_{2s} D_{2s}$ two properties can be derived:

$$\tilde{D}^* B^* B \tilde{D} = (I_3 \otimes P^{0,2s})^* \tilde{D}^* B^* B \tilde{D} (I_3 \otimes P^{0,2s})$$
$$\tilde{D}^* B^* B \tilde{D} = (I_3 \otimes P^{2s,0})^* \tilde{D}^* B^* B \tilde{D} (I_3 \otimes P^{2s,0})$$

Provided $M_i(u_i + 2s, v_i) = (P^{2s,0}) M_i(u_i, v_i), M_i(u_i, v_i + 2s) = (P^{0,2s}) M_i(u_i, v_i)$ the theorem is proven. □

For some motions $u_1, v_1, \cdots, u_k, v_k$, certain non-trivial compositions of transforms $\phi_1, ..., \phi_5$ keep the system invariant:

$$\hat{A} = (\phi_{i_1} \cdot \phi_{i_2} \cdot \cdots \cdot \phi_{i_m})(\hat{A}),$$

which makes it possible to express some rows of $\hat{A}$ by using elements from other rows. This is an additional resource for filter bank compression. The present authors have conducted a numeric experiment applying theorems 8 through 14 to compress filter banks. Compression ratios for filter size 16×16 and $k = 3$ input frames are summarized in Table 1.

Filter bank compression using symmetries

Problem	Original size	Compressed size	Compression factor
2D, s=2	16x[3x2x2x16x16]	26x16x16	7.38
2D, s=4	256x[3x4x4x16x16]	300x16x16	40.96
Bayer, s=2	256x[3x3x4x4x16x16]	450x16x16	81.92
Bayer, s=4	4096x[3x3x8x8x16x16]	25752x16x16	91.62

Conclusion

This paper has formulated a $L_2 - L_2$ joint demosaicing and super-resolution problem for reconstruction from multiple Bayer images and proposed a direct method to obtain a block diagonal form of the problem matrix. Closed-form transforms were provided. It is hoped that the presented proofs are clearer and easier to understand than the proofs provided in [2], [3], and [4].

Table 2 shows the computational complexity of finding the matrix inverse (marked "MI") for 1D, 2D and Bayer SR problems. Block diagonalization made it possible to reduce the complexity of 2D and Bayer SR problems from $\mathscr{O}(n^6)$ to $\mathscr{O}(n^2 s^4) + \mathscr{O}(n^2 \log n)$, where $n^2 \log n$ corresponds to the complexity of the block diagonalization process itself. Typically, n is much larger than s (as $n = 16, ..., 32, s = 2, ..., 4$) which provides significant economy.

This paper also studied the symmetries intrinsic to the Bayer SR problem. Theoretic analysis of these properties was conducted, and results of numeric simulations applied to filter bank

Complexity

Problem	Matrix size	Blocks	Block size	MI, original	MI, reduced
1D	$n \times n$	n/s	$s \times s$	n^3	ns^2
2D	$n^2 \times n^2$	$(n/s)^2$	$s^2 \times s^2$	n^6	$n^2 s^4$
Bayer	$3n^2 \times 3n^2$	$(n/(2s))^2$	$12s^2 \times 12s^2$	n^6	$n^2 s^4$

compression were provided. In the case of $k = 3, s = 4$, an $\times 80$ reduction of the filter bank has been obtained.

All the proofs are based on the apparatus of multilevel matrices and can easily be extended to deblurring, multi-frame deblurring, demosaicing, multi-frame demosaicing or de-interlacing problems.

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