A Geometric Foundation of Colorimetry

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Abstract

The physical properties of color are usually described by their spectra, eigenvector expansions or low-dimensional descriptors such as RGB or CIE-Lab. In the first part of the paper we show that many of the traditional methods can be unified in a framework where color spectra are elements of an infinite-dimensional Hilbert space that are described by projections onto low-dimensional spaces. We derive some fundamental geometrical properties of the subset of the Hilbert space formed by all color spectra. We describe the projection operators that map the elements of the Hilbert space to elements in a finite dimensional vector space. This leads to a generalization of the concepts of spectral locus and purple line. It will be shown that for geometrical reasons the color space is topologically equivalent to a cone. In the second part of the paper we illustrate the theoretical concepts with four large databases of spectra from color systems and a series of multi-spectral images of natural scenes. We verify the conical property of color space for these databases and compute their, geometrically defined, spectral locus and chromaticity properties. In the last section we relate the natural co-ordinate system in the conical color space to the traditional polar co-ordinates in CIE-Lab. We show that there is a good agreement between the geometrically defined hue-variable and the angular part of the polar co-ordinate system in CIE-Lab. There is also a clear correlation between the geometrical and the CIE-Lab saturation descriptors.

1. Introduction

Traditionally colors are either described by their spectral distribution functions $f(\lambda)$ (where λ is the wavelength) or by three-dimensional vectors in coordinate spaces like CIEXYZ, CIELab, RGB etc.. The spectral description of color is complete in the sense that it contains all information about the physical properties of the color. It is however difficult to handle in practice and it gives no explicit description of the intuitive properties of the color. The three-dimensional color coordinates on the other hand are designed to give a compact description of the color and they are often closely associated with special color devices such as monitors, computers, printers or human color vision. The restriction to only three coordinates is however too restrictive in many applications.

Apart from the three-dimensional color spaces and the spectral descriptions other systems based on eigenvector expansions have been used to describe colors (see [4, 10]). In this case the coefficients in the series expansion are used as color coordinates. The eigenvector description has the advantage that it is optimal in a minimum-least-squared-error sense, it gives a compact description of the color spectra and it is not limited to three dimensions. It has the drawback that its coefficients have no obvious intuitive interpretation.

In this paper we will put all of these approaches into a unifying framework. By doing so we will show that color spaces are convex sets in infinite-dimensional Hilbert spaces which project to a conical set under all reasonable projection operators. This conical structure will be verified for a number of spectral databases and for multi-spectral images of real-world scenes. We showed earlier that this conical structure has profound consequences for further processing in, for example, computer vision and pattern recognition [6, 8, 9]. In contrast to these earlier investigations were we motivated the conical structure by the non-negativity of color spectra we give here a more precise description of the origin and the structure of the cone.

Since the color coordinates are all located in a cone it is natural to describe the conical structure by the axis of the cone and a plane perpendicular to this axis. For the color description this means that we separate the intensity and the chromaticity of a color. This leads to the introduction of perspective projections which projects the full color coordinate vector to its chromaticity vector. Under the perspective projection the color cone is projected to a two-dimensional set which is topologically equivalent to a disk. We will therefore call it the chromaticity disk. We also show that the boundary of the chromaticity disk is pointwise fixed under illumination changes and the effect of illumination changes is therefore a deformation of the inner geometry of the chromaticity disk.

Many of the underlying ideas in this paper can be found in [5]. Here we connect them to our earlier work on conical color spaces, we verify their applicability with the help of various databases of spectral data and illustrate the empirical properties of some of the models.

The conical structure of the color cone has been in-

vestigated before (see [1, 11, 12]). These approaches are however all aimed at an understanding of human color perception. They thus start from some basic properties of the human color vision system and derive from them the geometrical properties of the space of perceived colors. In this paper we are not primarily interested in the space of perceived colors but in the space of spectral distributions of light. We will derive some of the properties of this space which are independent of the sensor system used to measure the spectral distributions. We will assume that the sensor system defines a linear projection of the space of spectral distributions and will have to require that this projection operator preserves convexity. Human color perception enters our investigation only when we demonstrate how well human color perception matches the properties of the mathematical operators introduced in this paper. We will not discuss the question were such a correspondence comes from but we want to mention that these findings seem to confirm the theory that there is a close connection between the statistics of natural scenes and the properties of human perception.

2. Color stimuli space

In an abstract mathematical context the vector space, the traditional linear color spaces and the eigenvector based descriptions of color spectra can be investigated in the following framework: The color stimulus (the radiation which enters the sensor) is an element $f(\lambda)$ in the Hilbert space $\mathbf{H}(I)$ of square integrable functions defined on the interval $I = [\lambda_{min}, \lambda_{max}]$. Here $\lambda_{min}, \lambda_{max}$ are the shortest and the longest wavelength that have an effect on the sensor. A typical interval is given by $I = [380nm, 800nm]$. Next a finite dimensional subspace $\mathbf{H}_N(I)$ of $\mathbf{H}(I)$ is selected. In this subspace a basis $b_n(\lambda), n = 0, \ldots N$ is introduced and the elements $f \in \mathbf{H}(I)$ are approximated by their projections $\mathbf{O}f$ onto $\mathbf{H}_N(I)$:

$$f(\lambda) \approx \mathbf{O}f(\lambda) = \tilde{f}(\lambda) = \sum_{n=0..N} \beta_n b_n(\lambda) \qquad (1)$$

The basis vectors are not necessary orthogonal or of unit length. For an orthonormal basis the coefficients are computed as $\beta_n = \langle f, b_n \rangle$ where $\langle f_1, f_2 \rangle$ is the scalar product in the Hilbert space.

For the finite-dimensional vector space description the basis functions $b_n(\lambda)$ are the Dirac functions δ_n located at the sampling points in the interval I. For the traditional color spaces such as CIEXYZ the basis functions are the color matching functions. In the eigenvector system the basis functions are the eigenvectors.

The set of all color stimuli $\mathbf{S}$ is obviously a proper subset of the complete Hilbert space $\mathbf{H}(I)$. Next we introduce the monochromatic color stimuli $m_{\lambda_0}(\lambda)$ as those spectral distributions that are concentrated at one given wavelength $\lambda_0 \in I$. They are given by the generalized function δ_{λ_0}. Sometimes we think of them as the functions $m_{\lambda_0}(\lambda_0) = 1$ and $m_{\lambda_0}(\lambda) = 0$ elsewhere. We introduce two special color stimuli in $\mathbf{S}$: the color black (defined by the function that is zero everywhere on I) and the equal energy stimulus $u(\lambda) = 1$ for all $\lambda \in I$ defining the color white. The half-line $\{cm_{\lambda_0}(\lambda_0), c \geq 0\}$ lies on the border of the set $\mathbf{S}$. This can be easily checked by observing that the line connecting the monochromatic stimulus $m_{\lambda_0}(\lambda)$ with the white point $u(\lambda)$ crosses the boundary of $\mathbf{S}$ at $m_{\lambda_0}(\lambda)$. From Grassmann's laws it follows that for two color stimuli $f_i(\lambda) \in \mathbf{S}$, $i \in \{1, 2\}$ the linear combinations $cf_1(\lambda) + (1 - c)f_2(\lambda), 0 \leq c \leq 1$ are also color stimuli. Thus the set $\mathbf{S}$ is convex and therefore the convex closure of the monochromatic stimuli.

The image $\mathbf{S_N} = \{\mathbf{O}f : f \in \mathbf{S}\}$ of $\mathbf{S}$ is a subset of the finite dimensional subspace $\mathbf{H}_N(I)$. The image of the set of monochromatic stimuli $\{m_{\lambda_0}(\lambda), \lambda_0 \in I\}$ forms a curve in $\mathbf{H}_N(I)$. Following traditional color science this curve is called the spectral locus. Since the monochromatic spectra lie on the boundary of $\mathbf{S}$ we will require that the spectral locus is on the boundary of $\mathbf{S_N}$. This is a non-trivial restriction on the projection operators $\mathbf{O}$. The line in $\mathbf{H}_N(I)$ which connects the projections of the endpoints $m_{\lambda_{min}}(\lambda)$ and $m_{\lambda_{max}}(\lambda)$ is called the purple line. We will often assume that the closed curve formed by the spectral locus and the purple line is topologically equivalent to a circle, ie. that it has no intersection points.

The color space $\mathbf{S}$ and its image $\mathbf{S_N}$ consist of half-lines originating in the color black. Both, $\mathbf{S}$ and $\mathbf{S_N}$, have therefore a natural decomposition as a direct product between the positive half-axis (related to the intensity of the colors) and a space we call the chromaticity space. Formally we define two elements f_1, f_2 in $\mathbf{H}(\mathbf{H}_N, \mathbf{S}$ or $\mathbf{S_N})$ as equivalent if $f_2 = cf_1$ for a positive number c. A point in the chromaticity space is an equivalence class of elements in the original space. We are primarily interested in the sets $\mathbf{S}, \mathbf{S_N}$ and for them we define the chromaticity spaces $\mathbf{C}, \mathbf{C_N}$ as the equivalence classes formed by their elements. We thus have $\mathbf{S} = \mathbf{R}^+ \bigoplus \mathbf{C}$ and $\mathbf{S_N} = \mathbf{R}^+ \bigoplus \mathbf{C_N}$ where $\mathbf{R}^+$ is the group of positive real numbers under multiplication. The projection from $\mathbf{S}$ to $\mathbf{C}$ defines a perspective projection operator $\mathbf{P}$ which cancels the influence of the intensity parameter and maps a color vector to its chromaticity description.

Next we observe that a color stimulus is formed by the interaction of the illumination light $l(\lambda)$ and the object reflectance function $o(\lambda)$. The reflectance function $o(\lambda)$ defines how much of the incoming light of wavelength λ is reflected from an object point. Its values are between zero and one: $0 \leq o(\lambda) \leq 1$. We exclude effects like fluorescence and assume that the stimulus is given by the product

of the illumination and the reflectance: $f(\lambda) = l(\lambda)o(\lambda)$. For a given stimulus the factorization into illumination and reflectance is not unique since a multiplication of l by a positive factor $c > 0$ and o by its inverse $1/c$ does not change the stimulus. For the orthogonal projection $\mathbf{O}$ from $\mathbf{H}(I)$ to $\mathbf{H}_N(I)$ we find that

$$\mathbf{O}(cf)(\lambda) = \mathbf{O}((cl)(\lambda)o(\lambda)) = \mathbf{O}(l(\lambda)(co)(\lambda))$$

Therefore we combine the orthogonal projection $\mathbf{O}$ is with a perspective projection $\mathbf{P}$. An example of such a projection is $\mathbf{a} = \mathbf{P}b$ with vector elements (see Eq.(1)):

$$\alpha_n = \frac{\beta_n}{\beta_0} \tag{2}$$

The general projection is obtained by $\mathbf{a} = \mathbf{PM}b$ where $\mathbf{M}$ is a non-singular matrix of size $(N+1) \times (N+1)$ and $\mathbf{P}$ is the special projection defined in Eq.(2). An example is the three-dimensional color space defined by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. It leads to the intensity value $r+g+b$ and chromaticity vector $(r/(r+g+b), g/(r+g+b))$ where we used r, g, b instead of β_n for $n = 0, 1, 2$.

For a monochromatic object reflectance function o at wavelength λ_0 wave have: $o(\lambda) = cm_{\lambda_0}(\lambda)$ and

$$f(\lambda) = l(\lambda)o(\lambda) = cl(\lambda)m_{\lambda_0}(\lambda) = cl(\lambda_0)m_{\lambda_0}(\lambda).$$

But $cl(\lambda_0)$ is a number and we see that the chromaticity $\mathbf{PO}f = \mathbf{PO}(lo)$ is independent of l. This shows that the spectral locus is independent of the illumination characteristics.

The spectral locus defines together with the purple line a closed curve. For a given point $\mathbf{w}$ (in the following called the white point) in the interior of this closed curve we define the polar co-ordinates (ρ, φ) for a selected point $\mathbf{x}$ inside or on the curve as follows: the radius is the euclidean distance between $\mathbf{x}$ and $\mathbf{w}$. The angular coordinate φ is the angle between the line connecting $\mathbf{x}$ and $\mathbf{w}$ and the positive x-axis. In the following we will assume that the spectral locus and the purple line define a curve which is topologically equivalent to a circle. For a point $\mathbf{b}$ on the spectral locus or the purple line with coordinates $(\rho(\mathbf{b}), \varphi(\mathbf{b}))$ we define: $R(\varphi) = R(\varphi(\mathbf{b})) = \rho(\mathbf{b})$. This defines a function on the unit circle. For a point inside the spectral locus with coordinates (ρ, φ) we define the normalized polar coordinates as $(\rho/R(\varphi), \varphi) = (r, \varphi)$. This construction depends only on the selected white-point and maps the spectral locus and the purple line to the unit circle. The operator which maps the chromaticity coordinates to the unit circle which depends only on the white point will be denoted by

$$\mathbf{C}_\mathbf{w} : \mathbf{x} = (\rho, \varphi) \mapsto (r, \varphi) \tag{3}$$

The complete mapping from the Hilbert space to the unit circle is thus given by $\mathbf{C}_\mathbf{w}\mathbf{PMO}$ where the matrix $\mathbf{M}$ defines the chromaticity plane and $\mathbf{w}$ the center of the chromaticity circle.

Up to now we have not introduced metric structures in the spaces considered so far. Since all vectors are located in cones we can consider them as elements in the usual euclidean spaces or as elements in conical spaces. It is therefore natural to measure their length with either the euclidean or the hyperbolic norm. For a stimulus with a coordinate vector $\beta = \begin{pmatrix} \beta_0 & \beta_1 & \dots & \beta_N \end{pmatrix}$ this leads to:

$$\begin{aligned} \|\beta\|_e^2 &= \beta_0^2 + \beta_1^2 + \ldots + \beta_N^2 \\ \|\beta\|_h^2 &= \beta_0^2 - \beta_1^2 - \ldots - \beta_N^2 \end{aligned} \tag{4}$$

For the special case of three-dimensional coordinates $\{\beta_0, \beta_1, \beta_2\} = \rho\{\cosh\alpha, \sinh\alpha\cos\varphi, \sinh\alpha\cos\varphi\}$ we find the following relations:

$$\begin{aligned} \text{coordinates:} \quad & \rho(\cosh\alpha, \sinh\alpha\cos\varphi, \sinh\alpha\cos\varphi) \\ \text{chromaticity:} \quad & \tanh\alpha(\cos\varphi, \sin\varphi) \\ \text{euclidean norm:} \quad & \|\beta\|_e^2 = \rho^2 \cdot (2\cosh^2\alpha - 1) \\ \text{hyperbolic norm:} \quad & \|\beta\|_h^2 = \rho^2. \end{aligned}$$

3. The spectral databases

The results described so far are based on mathematical and geometrical arguments only. These arguments show that there are strong theoretical reasons in favor of a conical coordinate system in color space. In our experiments we used several large databases of color spectra and investigated how their properties fit into this framework. The main result was that eigenvector based color descriptions are good representatives of the class of conical color spaces and that the geometrical chromaticity coordinates have a strong correlation to the perceptually defined chromaticity coordinates in the CIE-Lab and CIE-Luv system.

In our experiments we used databases of spectra measured from the Munsell, the NCS and the Pantone color systems. We also investigated multispectral images of natural scenes and spectral measurements from the GOME satellite. The Munsell database consists of 1269 spectra measured from 380nm to 800nm at 1nm steps [7]. The spectra from the NCS color system were collected in two databases. The older database consists of 1513 samples measured from 380nm to 780nm at 5nm intervals. The newer collection consists of 1750 spectra measured with 10nm increments between 400nm and 700nm. In some experiments we combined the old NCS database and the Munsell database to one large database with spectra defined in the range 380nm to 780nm sampled in 5nm steps. The Pantone database has 922 color spectra measured in 1nm increments between 380 and 780nm.

For each of these databases we restricted the wavelength range to 400nm to 700nm. Then we computed the

eigenvectors for each of the systems. The first three eigenvectors define an orthogonal projection operator to a three-dimensional space. Simple perspective projection as defined in Eq.(2) is then used to project the three-dimensional vectors to a plane. The location of the spectral locus for the four databases is shown in Figure 1. In the Figure we also show the location of the white-points of these databases where the white-point is defined by the equal energy spectral distribution $u(\lambda)$.

For all the databases we computed the projection of the spectra to the chromaticity disk. The Figure 2 shows the location of the spectra in the new NCS database in the chromaticity disk defined by the chromaticity plane given by the unit matrix $\mathbf{M}$ and the white point defined by the equal energy spectrum u.

4. Multi-spectral images

The second source of color spectra comes from 22 multi-spectral images, 10 taken in coral reefs and 12 in forests (for a detailed description of the images see [2, 3]). From these images we computed two eigenvector systems: one from the coral and one from the forest images. We then described the spectra from these images in both the Munsell-NCS coordinate system and in the system computed from the spectra in the same class.

For the forrest images we find that only three spectra lie outside the cone. For the underwater images we found that around 200 (out of 10×128^2) spectra are located outside the unit cone. Most of them are from the images horshe5 and horshe30 and we found that many of these outliers have a very low intensity. The results for the cones defined by the NCS-Munsell system and defined by the spectra in the same class were comparable. We assume that many of the spectra outside the cone may be effected by noise during the measurement process.

The chromaticity diagrams with the spectral locus and the white point for the coral database is shown in Figures 3.

5. GOME Spectrometer Data

The Global Ozone Monitoring Experiment (GOME) was launched on April 21st 1995. It is a spectrometer that measures the solar radiation scattered by the atmosphere in the spectral region from 240 to 790 nm with a spectral resolution of 0.2 to 0.4 nm, sensed by four individual linear detector arrays each with 1024 detector pixels. The field of view may be varied from 40 km along track and 320 km across track to 40 km along track and 40 km across track. In our experiments we extracted data from two detectors and interpolated the measurements so that the final spectra represented the satellite data in the range 407nm to 794nm with 1nm sampling. In the conversion we rejected spectra which were obviously flawed (for example spectra with negative values or spectra with very low measurement values). In this way we collected 59975 spectra from 31 original files. For these spectra we computed the eigenvectors, the spectral locus and the coordinates of the spectra in the first three eigenvector coordinate axis. We found that 619 spectra were located outside the cone. An inspection of these outliers showed that almost all of them came from the same file. The exceptional spectra in this file were all measured at consecutive time steps and for all of them the data in the third band had very low values whereas the data vectors from the fourth band had normal values. We therefore concluded that these spectra were obtained while one of the sensors did not work properly. The spectral locus and the white point for this database is shown in Figure 4

6. Relation to CIELab co-ordinates

One of the most important coordinate systems in color science is the CIE-Lab system where L describes the brightness of the color spectrum and the two-dimensional vector (a,b) its chromatic properties. Instead of using the (a,b) coordinates directly one can use polar coordinates (p, ψ) in the (a,b)-plane. The values of p and ψ indicate the saturation and the hue of the color spectrum. These coordinates are designed to mirror the properties of human color vision and are highly non-linear functions of the spectra. It is therefore interesting to see how they relate to the projection based measures r, φ as defined in equation (3). In Figure (5) the values of the Lab-hues ψ and the corresponding values of φ are illustrated. The comparison between the saturation variables p in Lab-space and r in the projections is found in Figure 6. These figures show that there is a good correspondence between the hue variables and a strong correlation between the saturation descriptors.

7. Conclusions

We derived a purely geometrical description of color spectra. This description is derived by a combination of an orthogonal and a perspective projection. The resulting description is closely related to traditional color systems that describe colors in terms of intensity, hue and saturation. Experimental studies show that this framework holds for a wide variety of color spectra.

Acknowledgements

The databases with the Munsell- and the Pantone spectra were supplied by the Computer Science and Physics Departments at the University of Joensuu, Finland. The NCS

databases were obtained from the Scandinavian Color Institute and the multi-spectral images are described in [2, 3]. The GOME spectra were supplied by ESA under contract number 1098.

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8. References

References

[1] M.H. Brill and E.B. Barrett. Closed-form extension of the anharmonic ratio to n-space. *Computer Vision, Graphics, and Image Processing*, 23:92–98, 1983.

[2] C.C. Chiao, T.W. Cronin, and D. Osoria. Color signals in natural scenes: characteristics of reflectance spectra and effects of natural illuminants. *J. Opt. Soc. Am. A*, 17(2):218–224, Feb. 2000.

[3] C.C. Chiao, D. Osoria, M. Vorobyev, and T.W. Cronin. Characterization of natural illuminants in forests and the use of digital video data to reconstruct illuminant spectra. *J. Opt. Soc. Am. A*, 17(10):1713–1721, Oct. 2000.

[4] J. Cohen. Dependency of the spectral reflectance curves of the Munsell color chips. *Psychon Science*, 1:369–370, 1964.

[5] J. J. Koenderink and Andrea J. van Doorn. The structure of colorimetry. In *Proc. AFPAC 2000*, pages 69–77. Springer, Lec. Notes Computer Science, 1888, 2000.

[6] R. Lenz and P. Meer. Non-euclidean structure of spectral color space. In E. A. Marszalec and E. Trucco, editors, *Polarization and Color Techniques in Industrial inspection*, volume 3826 of *Proceedings Europto Series*, pages 101–112. SPIE, 1999.

[7] R. Lenz, M. Österberg, J. Hiltunen, T. Jaaskelainen, and J. Parkkinen. Unsupervised filtering of color spectra. *Journal of the Optical Society of America A*, 13(7):1315–1324, 1996.

[8] Reiner Lenz. Color edge detectors for conical color spaces. In *Proc. International Conference on Color in Graphics an Image Processing*, pages 284–289, 2000.

[9] Reiner Lenz. Estimation of illumination characteristics. *IEEE Transactions Image Processing*, 10(7):1031–1038, July 2001.

[10] J.P.S. Parkkinen, J. Hallikainen, and T. Jaaskelainen. Characteristic spectra of Munsell colors. *Journal of the Optical Society of America A*, 6(2):318–322, 1989.

[11] Howard L. Resnikoff. Differantial geometry and color perception. *Journal Math. Biology*, 1:97–131, 1974.

[12] H. Yilmaz. *On Color Perception*, volume XX of *Int. School Physics, Enrico Fermi*, pages 239–251. Academic Press, 1962.

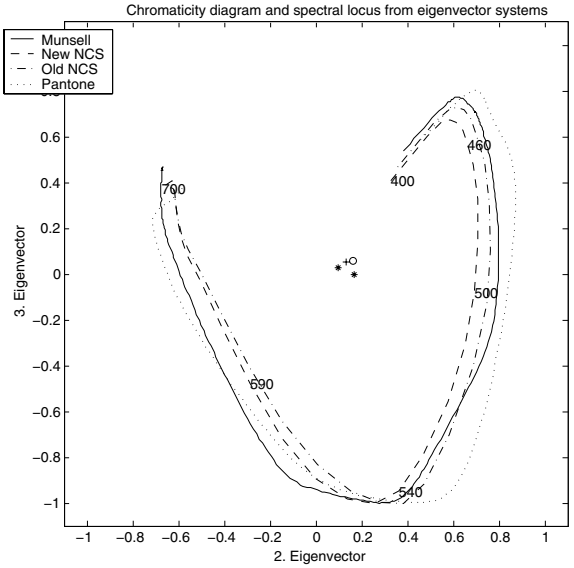

Figure 1: *Spectral locus and white points for the four databases*

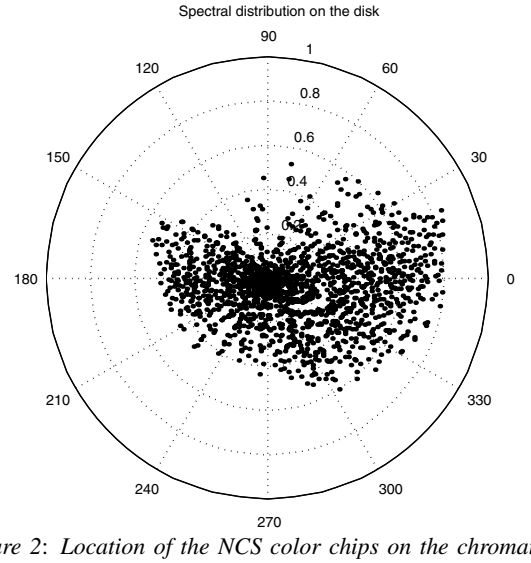

Figure 2: *Location of the NCS color chips on the chromaticity disk*

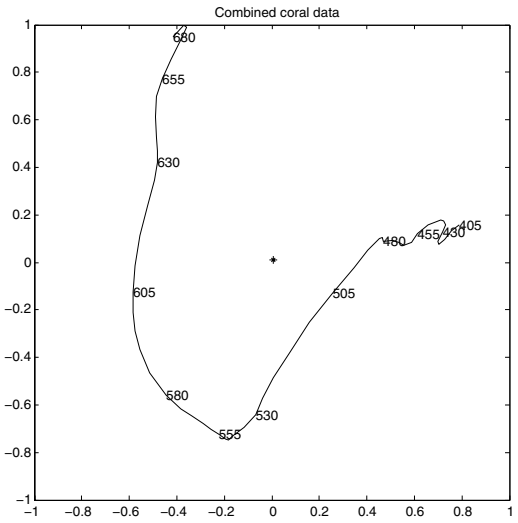

Figure 3: *Spectral locus and white point for the coral images*

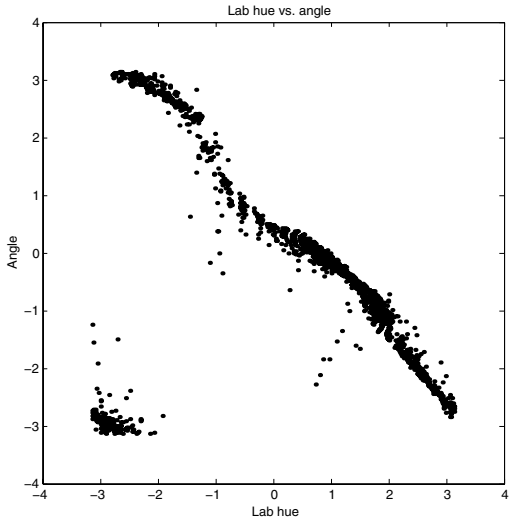

Figure 5: *Lab-hue vs. angular variable in projection*

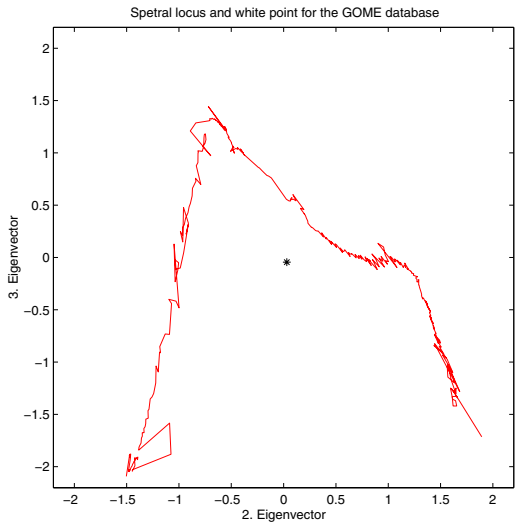

Figure 4: *Spectral locus and white point for the GOME database*

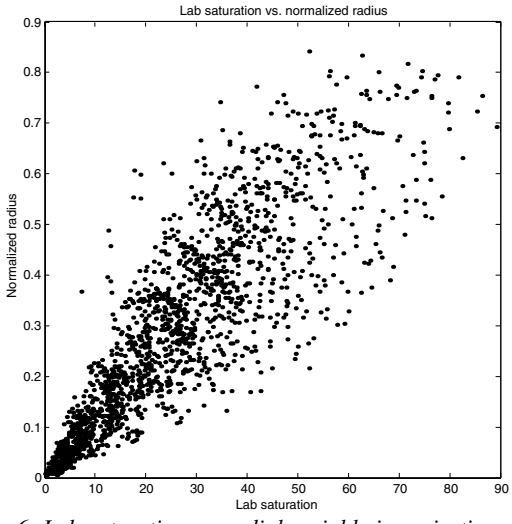

Figure 6: *Lab-saturation vs. radial variable in projection*