Orthogonal Illuminant Model and Its Application to Counting Metamers

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Abstract

This paper discusses about the most different illuminant (termed the orthogonal illuminant) compared with a reference illuminant. This is the first report on the definition of the orthogonal illuminant and on the solution of the optimization problem. As an application, the orthogonal illuminant is applied to counting metamers problem.

1.Introduction

This paper discusses about the most different illuminant (termed the orthogonal illuminant) compared with a reference illuminant based on metameric color-mismatch volume.

Metameric colors (metamers) are color stimuli with the same tristimulus values but different spectral radiation power distributions. One of the most important applications of a set of metamers generated with respect to a given illuminant and observer is to the determination of the magnitude of the color mismatches that will occur when the illuminant or the observer is changes. There have been studies[1)] of the boundaries of mismatches of metamers by N. Ohta and G. Wyszecki, but no reports have appeared about which illuminant yields the largest magnitude of color-mismatch volume. The illuminant which yields the largest color-mismatch volume is termed the orthogonal illuminant. The color-mismatch volume corresponds to the degree of difference.

This is the first report defining of the orthogonal illuminant and the solution to the optimization problem. The main subjects of this paper are the definition of the orthogonal illuminant and the derivation of the orthogonal illuminant by solving the optimization problem whose cost function is the volume of the color mismatch. Mismatch coordinates are known to form a closed solid in a color space. Linear programming is employed to calculate the volume of the solid. It is difficult to derive a solution that maximizes the volume of the solid by analytical methods. Hence, a search method for optimization called simulated annealing[2)] is employed.

In experiments, the orthogonal illuminant is derived for an illuminant of the completely flat spectrum: the ideal white illuminant. As an application, the orthogonal illuminant is applied to counting metamers problem and the experimental result is provided.

2.Orthogonal Illuminant Model and Its Application

2.1.Orthgonal Illuminant Model

Two objects with different spectral reflectance functions $\rho(\lambda)$ and $\rho'(\lambda)$ give rise to metamer stimuli when illuminated by $S(\lambda)$ if their corresponding tristimulus values X,Y,Z and X',Y',Z' are equal as follows:

$$
\begin{aligned}
&\sum_\lambda s(\lambda)\rho(\lambda)\bar{x}(\lambda) / \sum_\lambda s(\lambda)\bar{y}(\lambda) = \sum_\lambda s(\lambda)\rho'(\lambda)\bar{x}(\lambda) / \sum_\lambda s(\lambda)\bar{y}(\lambda) , \\
&\sum_\lambda s(\lambda)\rho(\lambda)\bar{y}(\lambda) / \sum_\lambda s(\lambda)\bar{y}(\lambda) = \sum_\lambda s(\lambda)\rho'(\lambda)\bar{y}(\lambda) / \sum_\lambda s(\lambda)\bar{y}(\lambda) , \\
&\sum_\lambda s(\lambda)\rho(\lambda)\bar{z}(\lambda) / \sum_\lambda s(\lambda)\bar{y}(\lambda) = \sum_\lambda s(\lambda)\rho'(\lambda)\bar{z}(\lambda) / \sum_\lambda s(\lambda)\bar{y}(\lambda) .
\end{aligned}
\tag{1}
$$

where

λ : wavelength.

Under the first illuminant (reference illuminant), the metameric match is described as follows, where $\left(X^{(1)},Y^{(1)},Z^{(1)}\right)$ is the coordinate of the metameric match for different values of $\rho(\lambda)$,

$$
\begin{aligned}
X^{(1)} &= \sum_{\lambda} s^{(1)}(\lambda)\rho(\lambda)\bar{x}(\lambda) / \sum_{\lambda} s^{(1)}(\lambda)\bar{y}(\lambda) \\
&= \sum_{\lambda} s^{(1)}(\lambda)\rho'(\lambda)\bar{x}(\lambda) / \sum_{\lambda} s^{(1)}\bar{y}(\lambda)\ , \\
Y^{(1)} &= \sum_{\lambda} s^{(1)}(\lambda)\rho(\lambda)\bar{y}(\lambda) / \sum_{\lambda} s^{(1)}(\lambda)\bar{y}(\lambda) \\
&= \sum_{\lambda} s^{(1)}(\lambda)\rho'(\lambda)\bar{y}(\lambda) / \sum_{\lambda} s^{(1)}\bar{y}(\lambda)\ , \\
Z^{(1)} &= \sum_{\lambda} s^{(1)}(\lambda)\rho(\lambda)\bar{z}(\lambda) / \sum_{\lambda} s^{(1)}(\lambda)\bar{y}(\lambda) \\
&= \sum_{\lambda} s^{(1)}(\lambda)\rho'(\lambda)\bar{z}(\lambda) / \sum_{\lambda} s^{(1)}\bar{y}(\lambda)\ ,
\end{aligned}
\tag{2}
$$

where

$$\rho(\lambda) \neq \rho'(\lambda).$$

When the illuminant is changed from the first one $S^{(1)}(\lambda)$ to the second $S^{(2)}(\lambda)$, the corresponding tristimulus values are given by,

$$
\begin{aligned}
X^{(2)} &= \sum_{\lambda} s^{(2)}(\lambda)\rho(\lambda)\bar{x}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ , \\
Y^{(2)} &= \sum_{\lambda} s^{(2)}(\lambda)\rho(\lambda)\bar{y}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ , \\
Z^{(2)} &= \sum_{\lambda} s^{(2)}(\lambda)\rho(\lambda)\bar{z}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ ,
\end{aligned}
\tag{3a}
$$

$$
\begin{aligned}
X^{(2)'} &= \sum_{\lambda} s^{(2)}(\lambda)\rho'(\lambda)\bar{x}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ , \\
Y^{(2)'} &= \sum_{\lambda} s^{(2)}(\lambda)\rho'(\lambda)\bar{y}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ , \\
Z^{(2)'} &= \sum_{\lambda} s^{(2)}(\lambda)\rho'(\lambda)\bar{z}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ .
\end{aligned}
\tag{3b}
$$

Here the metamerism is broken down and spread out,

$$\left(X^{(2)}, Y^{(2)}, Z^{(2)}\right) \neq \left(X^{(2)'}, Y^{(2)'} Z^{(2)'}\right).$$

It is known that the mismatch coordinates form a closed solid in a color space. We define the illuminant which makes the magnitude of the volume of the solid the largest against a reference illuminant as the orthogonal illuminant. Linear programming is employed to calculate the volume of the solid. It is difficult to derive the solution maximizing the volume of the solid by analytical methods. Hence, a search method for optimization called simulated annealing is employed.

The closed solid can be derived using the linear programming method in which eq.(2) and $0 \le \rho(\lambda) \le 1$ are the constraints, and eq.(3) is the objective function. The term ($X^{(1)}$, $Y^{(1)}$, $Z^{(1)}$) is a metameric color which is fixed, and for various values of $\rho(\lambda)$, ($X^{(2)}$, $Y^{(2)}$, $Z^{(2)}$) assumes mismatch values by changing the illuminant from $S^{(1)}(\lambda)$ to $S^{(2)}(\lambda)$.

[Linear programming formulation for the problem]

Constraints

$$0 \le \rho(\lambda) \le 1. \tag{4.a}$$

$$
\begin{aligned}
X^{(1)} &= \sum_{\lambda} s^{(1)}(\lambda)\rho(\lambda)\bar{x}(\lambda) / \sum_{\lambda} s^{(1)}(\lambda)\bar{y}(\lambda)\ , \\
Y^{(1)} &= \sum_{\lambda} s^{(1)}(\lambda)\rho(\lambda)\bar{y}(\lambda) / \sum_{\lambda} s^{(1)}(\lambda)\bar{y}(\lambda)\ , \\
Z^{(1)} &= \sum_{\lambda} s^{(1)}(\lambda)\rho(\lambda)\bar{z}(\lambda) / \sum_{\lambda} s^{(1)}(\lambda)\bar{y}(\lambda)\ .
\end{aligned}
\tag{4.b}
$$

Objective function

$$
\begin{aligned}
X^{(2)} &= \sum_{\lambda} s^{(2)}(\lambda)\rho(\lambda)\bar{x}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ , \\
Y^{(2)} &= \sum_{\lambda} s^{(2)}(\lambda)\rho(\lambda)\bar{y}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ , \\
Z^{(2)} &= \sum_{\lambda} s^{(2)}(\lambda)\rho(\lambda)\bar{z}(\lambda) / \sum_{\lambda} s^{(2)}(\lambda)\bar{y}(\lambda)\ .
\end{aligned}
\tag{5}
$$

The volume of the closed solid corresponds to a degree of difference between the first illuminant and the second illuminant, and we optimize the second illuminant maximizing the volume.

The optimized illuminant is the orthogonal illuminant.

2.2.Application

As an application, the optimized solution is applied to counting metamers problem. For counting metamers, different illuminants and different spectral reflectance should be considered. The effects of illuminants are considered in the color solid spanned by a pair of orthogonal illuminants, and different spectral reflectance are considered in the parameters of $\rho(\lambda)$ in eq.(2) and eq.(3).

3.Solution

We optimize the second illuminant maximizing the volume using simulated annealing.[2] In simulated annealing, $S^{(2)}(\lambda_i)$, $(i = 1, 2, \Lambda, n)$ quantized in the spectrum range are *n* dimensional parameters to be optimized, where *n* indicates the number of spectrum values. In simulated annealing process, reconfiguration of parameters $S^{(2)}(\lambda_i)$, $(i = 1, 2, \Lambda, n)$ is performed and for each reconfiguration acceptance or nonacceptance is determined. In simulated annealing, the reconfiguration and determination of acceptance are repeated, and the final state of the reconfiguration is the optimized solution. The following function ΔV is defined for the judgement of acceptance or nonacceptance of a reconfiguration.

$$
\begin{aligned}
\Delta V = &\ (V \text{ value after reconfiguration}) \\
&-(V \text{ value before reconfiguration}),
\end{aligned}
\tag{6}
$$

where V indicates the volume of a mismatch color solid calculated using linear programming described previously.

If ΔV increases, the reconfiguration is accepted. If ΔV decreases, the reconfiguration is accepted based on the probability of $p_a = \exp(-\Delta V / T)$ and rejected based on the probability of $p_r = 1 - \exp(-\Delta V / T)$, where T indicates the temperature of the annealing process. The larger the value of T, in other words, the higher the temperature, the more easily the reconfiguration is accepted; the smaller the value of T, in other words, the lower the temperature, the more difficult is the acceptance of reconfiguration. These are simulations of the annealing process. The reconfiguration of parameters is performed in descending the temperature using ΔV and the probability distribution as a reference. Local minima can be avoided using a probability distribution, and with decreasing temperature, the global optimum can be attained. The T value is reduced in decrements of ΔT down to 0. The reduction of ΔT is performed when the variation of the cost function V is the noise : the equilibrium state. In the repetition of the reconfigurations as temperature decreases, when the temperature reaches 0, the reconfigured state of $S^{(2)}(\lambda_i)$, $(i = 1,2,\Lambda, n)$ is the final optimized solution. We can obtain the solution of the orthogonal illuminant by this procedure.

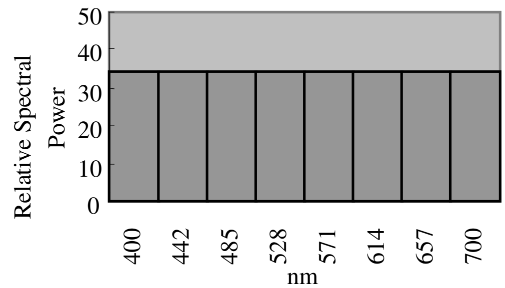
Simulated annealing is also applied to the application described in section 2.2. In the application, reconfiguration of $\rho(\lambda_i)$, $(i = 1,2,\Lambda, n)$ are performed to minimize the distance between tristimurus coordinates of an orthogonal illuminant pair.

The counting metamers problem is formulated as follows: How many or what proportion of the objective-color stimuli that belong to a given collection are approximately metameric with a given color stimulus of L_n^*, a_n^*, b_n^*. Approximately metameric means that the tristimulus values of the selected object-color stimuli should lie in a three-dimensional interval $\Delta L^*, \Delta a^*, \Delta b^*$ centered on L_n^*, a_n^*, b_n^*, the tolerances $\Delta L^*, \Delta a^*, \Delta b^*$ defining the closeness of metamerism demanded.

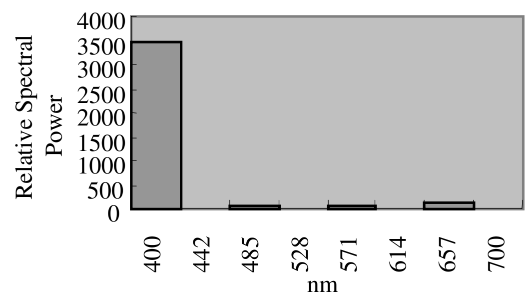
4.Experiments

Experiments deriving the orthogonal illuminants were performed. In the experiments, an illuminant of the completely flat spectrum was employed as the reference illuminant. The metameric match was on the x, y color coordinate of the illuminant. In these experiments, the wavelength range 400nm-700nm was divided into eight sections. In the simulated annealing process, the cost function was the mismatch volume. The initial temperature was $T = 1$ and $\Delta T = 1/10^4$. The reconfiguration based on $S^{(2)}(\lambda_i)$, $(i = 1,2,\Lambda, n)$ was performed using random numbers. The random numbers determined that which section i should be reconfigured and whether the modified value $\Delta S^{(2)}(\lambda_i)$ should be positive or negative. The step size of the modification was $|\Delta S^{(2)}(\lambda_i)| = 1/10^2$. The reconfiguration is performed keeping the Y value of the illuminant $S^{(2)}$ equals to the Y value of the illuminant $S^{(1)}$ (=100.0).

Figures 1 shows the experimental results. Figure 1(a) shows the reference illuminant and figure 1(b) shows its orthogonal illuminant. Against the completely flat spectrum, the pulse spectrum in the shortest wavelength division is dominant in the orthogonal illuminant which is reasonable in the mean of the orthogonal.



(a)



(b)

Figure 1 The reference illuminant and the orthogonal illuminant.

By using the resultant mismatch solid, the number N of metamers was obtained. The solid was mapped on the CIE $L^*a^*b^*$ space, and by the distance of $\Delta E = 0.7$ the solid was quantized, and counted. The result was $N = 1.8 \cdot 10^7$ for $\Delta E = 0.7$. The result is for all Y values of achromatic color.

5.Conclusions

In this paper, the orthogonal illuminant has been discussed. First the definition of the orthogonal illuminant has been described, and second the solution of the illuminant has been described. As an application, the orthogonal illuminant has been applied to counting metamers.

Experiments have been performed to derive the orthogonal illuminant. A illuminant of the completely flat spectrum has been employed as the ideal reference illuminant. The orthogonal illuminant had a dominant pulse spectrum which is reasonable as the orthogonal. Also experimental result of counting metamers has been indicated.

In this paper, there are not constraints on spectral reflectance and spectral distributions of illuminants. Hereafter, we will consider realistic constraints on spectral reflectance and spectral distributions of illuminants.

References

1. N. Ohta and G. Wyszecki, Theoretical Chromaticity-Mismatch Limits of Metamers Viewed Under Different Illuminants, J. Opt. Soc. Am., 65, p.327 (1975).
2. S. Kirkpatrick, Optimization by simulated annealing, J. Statist. Phys., 34, p.975 (1984).

Appendix

In this section, a characteristic is provided and proved to establish a rigid theoretical basis for the orthogonal illuminant. The features of the spectrum of the orthogonal illuminant are analyzed.

[Definition]

$\{e1,\ e2\}$ *indicates a set of coefficients of* $e1$, $e2$.

[Characteristic]

The spectrum shape of the orthogonal illuminant in Figure 1(b) is explained by means of an theoretical model.

[Proof]

Let assume s_o, s_s and ρ as follows :

$s_o(2\cdot n)$: value of the orthogonal illuminant spectrum at the index position of $2\cdot n$. Where $2\cdot n$ indicates the ($2\cdot n$)-th position in the wavelength range, and $0\le n\le 3$.

$s_o(2\cdot n+1)$: value of the orthogonal illuminant spectrum at the index position of $2\cdot n+1$. Where $2\cdot n+1$ indicates the ($2\cdot n+1$)-th position in the wavelength range, and $0\le n\le 3$.

$s_s(2\cdot n)$: value of an illuminant spectrum without 0 spectrum values at the index position of $2\cdot n$.

$s_s(2\cdot n+1)$: value of an illuminant spectrum without 0 spectrum values at the index position of $2\cdot n+1$.

$\rho(2\cdot n)$: value of a spectral reflectance at the index $2\cdot n$.
$\rho(2\cdot n+1)$: value of a spectral reflectance at the index $2\cdot n+1$.

The alternate appearance of 0 spectrums in the orthogonal illuminant can be described as follows :

$$s_o(2\cdot n)=\nu(2\cdot n) \qquad (A.1.a)$$

$$s_o(2\cdot n+1)=0. \qquad (A.1.b)$$

An illuminant spectrum without 0 spectrum values can be described as follows :

$$s_s(2\cdot n)=\nu_{c1}(2\cdot n)\ (\neq 0) \qquad (A.2.a)$$

$$s_s(2\cdot n+1)=\nu_{c2}(2\cdot n+1)\ (\neq 0). \qquad (A.2.a)$$

For relative spectrum power preservation between the two illuminants for each interval, the following relation should consists :

$$\nu(2\cdot n)=\nu_{c1}(2\cdot n)+\nu_{c2}(2\cdot n+1), \qquad (A.3)$$

where, the local average of $\nu_{c1}(2\cdot n)$ and $\nu_{c2}(2\cdot n+1)$ corresponds to the local average of $\nu(2\cdot n)$ and 0 in the same wavelength interval.

Let assume that $\{\rho_{c1},\rho_{c2}\}$ satisfies eq.(5), and under the assumption of the following equation :

$$\rho(2\cdot n)=\rho_{c1}(2\cdot n)+\rho_{c2}(2\cdot n+1), \qquad (A.4)$$

ρ satisfies eq.(5), approximately. Because for sufficiently small sampling intervals, continuous function of $s^{(1)}$ and the color matching functions $\bar{x},\ \bar{y},\ \bar{z}$ keep the relation of eq.(5) for both ρ and $\{\rho_{c1},\rho_{c2}\}$. For sufficiently small sampling, the relations below are consistent approximately:

$$s^{(1)}(2\cdot n)\cong s^{(1)}(2\cdot n+1) \qquad (A.5.a)$$

$$\begin{aligned}\bar{x}(2\cdot n)&\cong\bar{x}(2\cdot n+1),\\ \bar{y}(2\cdot n)&\cong\bar{y}(2\cdot n+1),\\ \bar{z}(2\cdot n)&\cong\bar{z}(2\cdot n+1).\end{aligned} \qquad (A.5.b)$$

Using the relation of eq.(A.5), the relation of eq.(5) consists approximately as follows :

$$\begin{aligned}
X^{(1)}&=\left(\sum_n s^{(1)}(2\cdot n)\rho_{c1}(2\cdot n)\bar{x}(2\cdot n)+\right.\\
&\quad\left.\sum_n s^{(1)}(2\cdot n+1)\rho_{c2}(2\cdot n+1)\bar{x}(2\cdot n+1)\right)\Big/\sum_n s^{(1)}(n)\bar{y}(n)\\
&\cong\sum_n s^{(1)}(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n)\Big/\sum_n s^{(1)}(2\cdot n)\bar{y}(2\cdot n),\\
Y^{(1)}&=\left(\sum_n s^{(1)}(2\cdot n)\rho_{c1}(2\cdot n)\bar{y}(2\cdot n)+\right.\\
&\quad\left.\sum_n s^{(1)}(2\cdot n+1)\rho_{c2}(2\cdot n+1)\bar{y}(2\cdot n+1)\right)\Big/\sum_n s^{(1)}(n)\bar{y}(n)\\
&\cong\sum_n s^{(1)}(2\cdot n)\rho(2\cdot n)\bar{y}(2\cdot n)\Big/\sum_n s^{(1)}(2\cdot n)\bar{y}(2\cdot n), \qquad (A.6)\\
Z^{(1)}&=\left(\sum_n s^{(1)}(2\cdot n)\rho_{c1}(2\cdot n)\bar{z}(2\cdot n)+\right.\\
&\quad\left.\sum_n s^{(1)}(2\cdot n+1)\rho_{c2}(2\cdot n+1)\bar{z}(2\cdot n+1)\right)\Big/\sum_n s^{(1)}(n)\bar{y}(n)\\
&\cong\sum_n s^{(1)}(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n)\Big/\sum_n s^{(1)}(2\cdot n)\bar{y}(2\cdot n).
\end{aligned}$$

Eq. (A.6) shows that both $\rho(2\cdot n)$ and $\{\rho_{c1}(2\cdot n), \rho_{c2}(2\cdot n+1)\}$ are satisfied in the relation of eq. (A.5). On the assumption of independence of color matching function vectors (which can be proved but omitted in this paper), its coefficients $\rho(2\cdot n)$, $(1\le n\le 3)$ are not correlated each other and $\{\rho(2\cdot n), 0\}$ is correlated only with $\{\rho_{c1}(2\cdot n), \rho_{c2}(2\cdot n+1)\}$ in the same interval. Hence, the formation of eq.(A.4) is correct satisfying eq.(5) for both sides of eq.(A.4).

The $X^{(2)}$ coordinate (not normalized) of the CIE XYZ space for the orthogonal illuminant is calculated as follows:

$$\begin{aligned} X^{(2)} &= \sum s_0\rho\bar{x} = \sum_n s_0(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n) \\ &+ \sum_n s_0(2\cdot n+1)\rho(2\cdot n+1)\bar{x}(2\cdot n+1) \qquad (A.7) \\ &= \sum_n v(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n). \end{aligned}$$

On the term $v(2\cdot n)\rho(2\cdot n)$ in eq.(A.7), the following relation is consistent by the expansion :

$$\begin{aligned} &\nu_{c1}(2\cdot n)\rho_{c1}(2\cdot n) + \nu_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1) \\ &\le \nu_{c1}(2\cdot n)\rho_{c1}(2\cdot n) + \nu_{c1}(2\cdot n+1)\rho_{c2}(2\cdot n+1) \\ &\quad + \nu_{c2}(2\cdot n)\rho_{c1}(2\cdot n) + \nu_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1) \qquad (A.8) \\ &= \nu(2\cdot n)\rho(2\cdot n). \end{aligned}$$

By applying eq.(A.8) to the right side of eq.(A.7) the following relation is derived :

$$\begin{aligned} &\sum_n (v_{c1}(2\cdot n)\rho_{c1}(2\cdot n) + v_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1))\bar{x}(2\cdot n) \\ &\le \sum_n v(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n). \end{aligned} \qquad (A.9)$$

The left side of eq.(A.9) is calculated on the sampling of $2n$, and the error ε between the calculations for each sampling point is as follows :

$$\begin{aligned} \varepsilon &= \sum_n v_{c1}(2\cdot n)\rho_{c1}(2\cdot n)\bar{x}(2\cdot n) \\ &+ \sum_n v_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1)\bar{x}(2\cdot n+1) \qquad (A.10) \\ &- \sum_n (v_{c1}(2\cdot n)\rho_{c1}(2\cdot n) + v_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1))\bar{x}(2\cdot n). \end{aligned}$$

Based on eq.(A.9), for sufficiently small ε, the following relation consists :

$$\begin{aligned} &\sum_n (v_{c1}(2\cdot n)\rho_{c1}(2\cdot n) + v_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1))\bar{x}(2\cdot n) + \varepsilon \\ &\le \sum_n v(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n). \qquad (A.11) \end{aligned}$$

Eq.(A.11) is rewritten as follows by using eq.(A.10),

$$\begin{aligned} &\sum_n \begin{pmatrix} v_{c1}(2\cdot n)\rho_{c1}(2\cdot n)\bar{x}(2\cdot n) + \\ v_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1)\bar{x}(2\cdot n+1) \end{pmatrix} \\ &\le \sum_n v(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n) \qquad (A.12) \end{aligned}$$

Furthermore, eq.(A.10) is converted as follows :

$$\varepsilon = \sum_n v_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1)(\bar{x}(2\cdot n+1) - \bar{x}(2\cdot n)) \qquad (A.13)$$

Eq.(A.10) indicates that if $\bar{x}(2\cdot n) \leftarrow \bar{x}(2\cdot n+1)$ then $0 \leftarrow \varepsilon$. This indicates that for sufficient small sampling intervals, ε is sufficiently close to 0, and the relation of eq.(A.12) consists.

For the reference illuminant used in the experiment, the relations of

$$\bar{y}(2\cdot n) \leftarrow \bar{y}(2\cdot n+1) \text{ and } v_{c1}(2\cdot n) \cong v_{c2}(2\cdot n+1)$$

are consist for sufficient small sampling intervals. Using these relations, the normalized relationship of eq.(A.12) is as follows :

$$\begin{aligned} &\sum_n v_{c1}(2\cdot n)\rho_{c1}(2\cdot n)\bar{x}(2\cdot n) / \sum_n v_{c1}(2\cdot n)\bar{y}(2\cdot n) \\ &+ \sum_n v_{c2}(2\cdot n+1)\rho_{c2}(2\cdot n+1)\bar{x}(2\cdot n+1) / \sum_n v_{c2}(2\cdot n+1)\bar{y}(2\cdot n+1) \\ &\le \sum_n v(2\cdot n)\rho(2\cdot n)\bar{x}(2\cdot n) / \sum_n v(2\cdot n)\bar{y}(2\cdot n) \qquad (A.14) \end{aligned}$$

Eq.(A.14) shows that the $X^{(2)}$ coordinate of the orthogonal illuminant including 0 spectrums is greater than the $X^{(2)}$ coordinate of the illuminant without 0 spectrum. The greater the $X^{(2)}$ value the larger the solid volume.

The same relations are $Y^{(2)} = \sum s_o\rho\bar{y}$

consistent in the same way for , $Z^{(2)} = \sum s_o\rho\bar{z}$.