Optimizing Color-Matching Functions for Individual Observers Using a Variation Method

Nobuhito Matsushiro*, Noboru Ohta, and Mark Q. Shaw**
Munsell Color Science Laboratory, Center for Imaging Science,
Rochester Institute of Technology,
Rochester, New York
****Visiting Scientist from Oki Data Corporation, Japan***
*****Applied Science Fiction, USA***

Abstract

This paper describes the optimization of the color matching functions (cmfs) of an individual observer based on metameric pairs using a variation method. This is a so much simplified method for estimating rough and ready cmfs of an individual observer in comparison with past experiments. Experiments have been performed using measured metamer spectral data.

1. Introduction

The cmfs of the standard observer are the fundamental basis of colorimetry. The CIE color matching functions (cmfs) were defined simply as an average of the functions of observers with normal color vision, and the standard colorimetric observer was defined as a hypothetical one that has the average cmfs so defined. Therefore, the cmfs of real observers with normal color vision do not agree exactly with those of the CIE standard colorimetric observer's.

This paper describes the optimization of the cmfs of an individual observer based on metameric pairs using a variation method.[1-7] This is a so much simplified method for estimating rough and ready cmfs of an individual observer in comparison with past experiments. The underlying assumption for the optimization is that the optimum cmfs will predict that the integrated cone responses of a metameric pair are equal. A feature of the optimization method is that the color difference in a metamer pair can be optimized to 0 at a boundary condition in the variation method, and the smoothness of the modified cmfs result from the cost function of the least mean square of modified values in the variation method.

The integrated cone responses of a metameric pair are not equal when an individual metameric pairs are evaluated by the CIE cmfs because of the difference between the CIE cmfs and those of the individual observer. Using the proposed optimization method, the color difference in the metamer data has been decreased to $\Delta E = 0$ with the modified cmfs predicting the cmfs of the individual observer.

2.Deriving optimal Color-Matching Functions

2.1. Basic Algorithm

Two objects with different spectral reflectance functions $\rho(\lambda)$ and $\rho'(\lambda)$ give rise to metamer stimuli when illuminated by $S(\lambda)$ if their corresponding tristimulus values X, Y, Z and X', Y', Z' are equal as follows:

$$\begin{aligned}\sum_{\lambda} \rho(\lambda) S(\lambda) \bar{x}(\lambda) \Delta\lambda &= \sum_{\lambda} \rho'(\lambda) S(\lambda) \bar{x}(\lambda) \Delta\lambda, \\ \sum_{\lambda} \rho(\lambda) S(\lambda) \bar{y}(\lambda) \Delta\lambda &= \sum_{\lambda} \rho'(\lambda) S(\lambda) \bar{y}(\lambda) \Delta\lambda, \\ \sum_{\lambda} \rho(\lambda) S(\lambda) \bar{z}(\lambda) \Delta\lambda &= \sum_{\lambda} \rho'(\lambda) S(\lambda) \bar{z}(\lambda) \Delta\lambda,\end{aligned} \quad (1)$$

The proposed method for optimizing the cmfs of an individual observer using a variation method is derived below.

For convenience, symbols are

$$\begin{gathered} X_1 = X,\ X_2 = Y,\ X_3 = Z, \\ q_1(\lambda) = \bar{x}(\lambda),\ q_2(\lambda) = \bar{y}(\lambda),\ q_3(\lambda) = \bar{z}(\lambda), \\ q(\lambda) = (q_1(\lambda), q_2(\lambda), q_3(\lambda)), \\ J(\lambda) = \rho(\lambda) S(\lambda). \end{gathered} \quad (2)$$

Let $q_i^*(\lambda), (i = 1,2,3)$ be the modified cmfs with a variation term $\Delta q_i(\lambda), (i = 1,2,3)$,

$$\begin{gathered} q_i^*(\lambda) = q_i(\lambda) + \Delta q_i(\lambda),\ (i = 1,2,3), \\ q_i^*(\lambda) = (q_1^*(\lambda), q_2^*(\lambda), q_3^*(\lambda))', \\ \Delta q(\lambda) = (\Delta q_1(\lambda), \Delta q_2(\lambda), \Delta q_3(\lambda)). \end{gathered} \quad (3)$$

Let $\rho_r(\lambda), \rho_m(\lambda)$ be a spectral reflectance of a reference data and that of a metamer data, respectively. The difference between the reference and the metamer object color stimuli is as follows:

$$\Delta J(\lambda) = \rho_r(\lambda) S(\lambda) - \rho_m(\lambda) S(\lambda) \quad (4)$$

The tristimulus values of $\Delta J(\lambda)$ related to $q_i^*(\lambda)$ are then given by

$$Q_i^* = (\Delta J(\lambda) \cdot q_i^*(\lambda)), \quad (i = 1,2,3). \quad (5)$$

Constraints are imposed as follows on eq.(5):

[Constraints on the variation method]

$$\Delta Q_i = (\Delta J(\lambda) \cdot q_i^*(\lambda)) = const_i \quad (given), \quad (i = 1,2,3). \quad (6)$$

[Cost function of the variation method]

$$CF(\Delta q) = \sum_\lambda cf(\Delta q(\lambda)) \Delta\lambda \quad (7)$$

where
$cf(\Delta q(\lambda))$: cost function for each λ. At $\Delta q = 0$ the minimum,
$CF(\Delta q)$: cost function for entire wave-length range.

[Constraint with unknown parameters of the variation method]

$$\sum_\lambda \left(\sum_{i=1}^{3} \mu_i \Delta J(\lambda) \cdot q_i^*(\lambda) \right) = const. \quad (8)$$

where

$\mu_i \ (i = 1,2,3)$, *:Lagrangian unknown parameters.*

The cf function is expanded by using the Taylor expansion as

$$cf_t(\Delta q(\lambda)) = cf(q(\lambda)) + \nabla cf(q(\lambda)) \Delta q(\lambda) + (1/2)(\Delta q(\lambda))^t H(q(\lambda)) \Delta q(\lambda), \quad (9)$$

where

$$\nabla cf(q(\lambda)) = \left(\frac{\partial cf(q)}{\partial q_1}, \frac{\partial cf(q)}{\partial q_2}, \frac{\partial cf(q)}{\partial q_3} \right)^t = (cf_{q1}(q), cf_{q2}(q), cf_{q3}(q))^t,$$

$$H(q) = \begin{bmatrix} \frac{\partial^2 cf(q)}{\partial q_1 \partial q_1} & \frac{\partial^2 cf(q)}{\partial q_1 \partial q_2} & \frac{\partial^2 cf(q)}{\partial q_1 \partial q_3} \\ \frac{\partial^2 cf(q)}{\partial q_2 \partial q_1} & \frac{\partial^2 cf(q)}{\partial q_2 \partial q_2} & \frac{\partial^2 cf(q)}{\partial q_2 \partial q_3} \\ \frac{\partial^2 cf(q)}{\partial q_3 \partial q_1} & \frac{\partial^2 cf(q)}{\partial q_3 \partial q_2} & \frac{\partial^2 cf(q)}{\partial q_3 \partial q_3} \end{bmatrix}, \quad \textit{Hessian matrix},$$

and the following Lagrange function is derived.

[Lagrange function of the variation method]

$$F = CF - 2 \cdot const = \sum_\lambda f \Delta\lambda \quad (10)$$

$$= \sum_\lambda \left[cf_t(\Delta q(\lambda)) - 2 \sum_{i=1}^{3} \mu_i \Delta J(\lambda) q_i^*(\lambda) \right] \Delta\lambda$$

$$= \sum_\lambda \left[cf(q(\lambda)) + \sum_{i=1}^{3} \Delta q_i(\lambda) cf_{qi}(q(\lambda)) + (1/2)(\Delta q(\lambda))^t H(q(\lambda)) \Delta q(\lambda) - 2 \sum_{i=1}^{3} \mu_i \Delta J(\lambda) \cdot q_i^*(\lambda) \right] \Delta\lambda$$

$$= \sum_\lambda \left[cf(q(\lambda)) + (1/2)(\Delta q_i(\lambda))^t H(q(\lambda)) \Delta q(\lambda) - 2 \sum_{i=1}^{3} (\mu_i \Delta J(\lambda) - (1/2) cf_{qi}(q(\lambda))) \Delta q_i(\lambda) - 2 \sum_{i=1}^{3} \mu_i \Delta J(\lambda) \cdot q_i(\lambda) \right] \Delta\lambda$$

$$= \sum_\lambda \left[cf(q(\lambda)) + (1/2)(\Delta q(\lambda))^t H(q(\lambda)) \Delta q(\lambda) - 2V^{(1)} \Delta q(\lambda) - 2V^{(2)} q(\lambda) \right] \Delta\lambda,$$

where
f: Lagrange function of the variation method for each λ,
F: Lagrange function of the variation method for all wavelengths,

$$\begin{aligned} V^{(1)}(\lambda) &= (V_1^{(1)}(\lambda), V_2^{(1)}, V_3^{(1)})^t \\ &= (\mu_1 \Delta J(\lambda) - cf_{q1}(q(\lambda))/2, \mu_2 \Delta J(\lambda) - cf_{q2}(q(\lambda))/2, \\ &\quad \mu_3 \Delta J(\lambda) - cf_{q3}(q(\lambda))/2), \\ V^{(2)}(\lambda) &= (V_1^{(2)}(\lambda), V_2^{(2)}, V_3^{(2)})^t \\ &= (\mu_1 \Delta J(\lambda), \mu_2 \Delta J(\lambda), \mu_3 \Delta J(\lambda)). \end{aligned} \quad (11)$$

A modified value Δq is independent of q and, based on eq.(11), the following equation is derived:

$$\Delta f / \Delta q = H(q(\lambda)) \Delta q(\lambda) - 2V^{(1)}(\lambda) = 0. \quad (12)$$

Equation (6) is converted as follows:

$$(\Delta J(\lambda) \cdot \Delta q_i(\lambda)) = const_i', \quad (13)$$

where

$$const_i' = const_i - (\Delta J(\lambda) \cdot q_i(\lambda)).$$

By eliminating $\Delta q_i(\lambda), (i = 1,2,3)$ in eqs.(12) and (13), a linear equation of the parameters $\mu_i, (i = 1,2,3)$ are derived. By solving the equation for the parameters $\mu_i, (i = 1,2,3)$ the Δq value and the modified cmfs are derived as follows:

$$\Delta q(\lambda) = 2H^{-1}(q(\lambda)) V^{(1)}(\lambda), \quad (14)$$

$$q_i^*(\lambda) = q_i(\lambda) + \Delta q_i(\lambda), \quad (i = 1,2,3) \quad (15)$$

The method derives the optimum solution over the entire wavelength range (see Appendix).

2.2.Description of the Cost Function

The cost function of the variation method is derived considering the CIE $L^*a^*b^*$ color space of the perception of color differences by the human visual system. The cost function measures the CIE $L^*a^*b^*$ sensitivity depending on variations in the CIE XYZ cmfs as follows:

$$g(r)=\begin{cases} r^{\frac{1}{3}}, & r>0.008856,\\ 7.787r+\dfrac{16}{116}, & r\le 0.008856.\end{cases}$$

$$cf(\Delta q(\lambda)) = cf_1(\Delta q(\lambda)) + cf_2(\Delta q(\lambda))$$

$$=\left\{(\Delta L)^2+(\Delta a)^2+(\Delta b)^2\right\}+\omega\left\{(\Delta q_1(\lambda))^2+(\Delta q_2(\lambda))^2+(\Delta q_3(\lambda))^2\right\} \quad (16)$$

where w *(0< w)* is a coefficient in the cost function. The first term $cf_1(\Delta q(\lambda))$ in eq.(16) is for the $L^*a^*b^*$ sensitivity and the second term $cf_2(\Delta q(\lambda))$ in eq.(16) is for the smoothness of the modified cmfs. Equation (16) is expanded using the Taylor expansion, and the following Hessian matrix of the cost function is derived :

$$H(q(\lambda))=\begin{bmatrix} H_{11} & H_{12} & H_{13}\\ H_{21} & H_{22} & H_{23}\\ H_{31} & H_{32} & H_{33}\end{bmatrix},$$

$$\begin{aligned}
H_{11} &= \left(\frac{500}{X_n}\right)^2\left(\frac{dg}{dr}\left(\frac{q_1(\lambda)}{X_n}\right)\right)^2 + w,\\
H_{12} &= -\frac{500^2}{X_nY_n}\frac{dg}{dr}\left(\frac{q_1(\lambda)}{X_n}\right)\frac{dg}{dr}\left(\frac{q_2(\lambda)}{Y_n}\right),\\
H_{13} &= 0,\\
H_{21} &= -\frac{500^2}{X_nY_n}\frac{dg}{dr}\left(\frac{q_1(\lambda)}{X_n}\right)\frac{dg}{dr}\left(\frac{q_2(\lambda)}{Y_n}\right)\\
H_{22} &= \frac{116^2+500^2+200^2}{Y_n^2}\left(\frac{dg}{dr}\left(\frac{q_2(\lambda)}{Y_n}\right)\right)^2+\omega,\\
H_{23} &= -\frac{200^2}{Y_nZ_n}\frac{dg}{dr}\left(\frac{q_2(\lambda)}{Y_n}\right)\frac{dg}{dr}\left(\frac{q_3(\lambda)}{Z_n}\right),\\
H_{31} &= 0,\\
H_{32} &= -\frac{200^2}{Y_nZ_n}\frac{dg}{dr}\left(\frac{q_2(\lambda)}{Y_n}\right)\frac{dg}{dr}\left(\frac{q_3(\lambda)}{Z_n}\right),\\
H_{33} &= \left(\frac{200}{Z_n}\right)^2\left(\frac{dg}{dr}\left(\frac{q_3(\lambda)}{Z_n}\right)\right)^2 + w.
\end{aligned} \quad (17)$$

The first derivative of the cost function is as follows :

$$\nabla cf(q(\lambda))=\left(\frac{\partial cf(q)}{\partial q_1},\frac{\partial cf(q)}{\partial q_2},\frac{\partial cf(q)}{\partial q_3}\right)^t=\left(cf_{q1}(q),cf_{q2}(q),cf_{q3}(q)\right)^t=(0,\ 0,\ 0). \quad (18)$$

Equations (17) and (18) are applied to the calculation of eq.(14).

The Hessian matrix (eq.(17)) is semipositive for any $q(\lambda)$, as shown in Characteristic in the Appendix, so the truncated cost function (eq.(9)) is convex based on Theorems 1 and 2 in the Appendix.

3. Experiments

3.1. Experimental Data Source

Shaw and Fairchild[8)9)] (1999) designed a visual experiment to allow observers to perform visual color matching between a neutral gray card of $L^*=50$ created with a Fujix Pictrography 3000 color printer, and an ACS VCS 10 additive mixing device. The viewing booth had both fluorescent daylight and incandescent illumination to view the colors. The seven discs in the ACS VCS 10 were white, red, green, blue, yellow, purple and black. The matching field was 8cm × 9cm, subtending a visual angle of 7°. Observers were seated 30 inches from the stimuli and asked to make an exact match to the gray card using only the three primaries specified. When a color match had been achieved, the PhotoResearch PR650 was used to measure the spectral radiance of the metamer from the observer's angle of view. Each observer was asked to repeat the experiment 10 times. In this experiments of our paper, matamer data of two observers are employed.

3.2. Optimization of cmfs

In optimizing cmfs, the metamer data in section 3.1. were employed. The experimental data were within a common wavelength range of 400nm-700nm (in 5nm steps). The spectrum data for each observer were averaged to reduce experimental noise. In the experiments, the cmfs of the CIE 1931 standard colorimetric observer were used as the standard reference to derive the modified cmfs optimized to an individual observer. The cost function of section 2.2. was employed. The weighting coefficient was $\omega=10^2$.

Figure 1, 2 shows the modified cmfs for observers 1, 2, respectively. In the optimization of Figure 1, 2, the constraint of eq.(6) was imposed on the variation method and $const_i=0,\ (i=1,2,3)$ for $\Delta E=0$. Figure 1 is almost the same as the CIE 1931 cmf. In Figure 2, except for the change in the range from 550nm-700nm in the modified $\bar{z}(\lambda)$ function, the modified cmfs are smooth and realistic. As for the smoothness of the modified cmfs, the least mean square does not necessarily ensure the continuity of the first-order derivative, although it is a general constraint for smoothness.

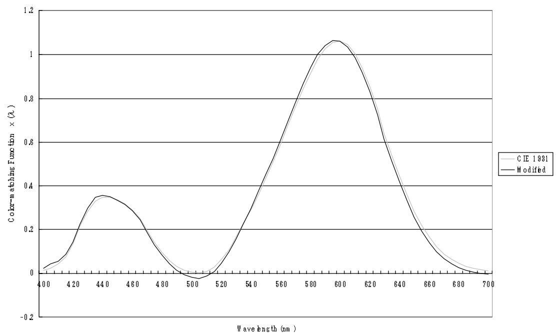
3.3. Discussion

In optimization problems, there is a difficulty in distinguishing between experimental error and the optimization error. In the proposed optimization method, $\Delta E=0$ is certain and thus, the optimization error can be neglected and only experimental error is included in the results. This technique can be repeated across a number of metameric matches to obtain a good statistical estimate of an individual observer's cmf. The solution of the method has an unbiased property against the cmfs of an individual observer based on expectations (see Appendix).

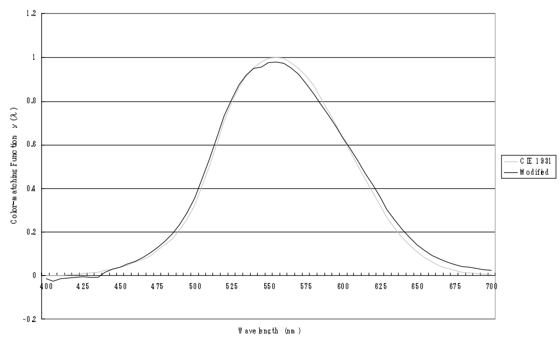
4. Conclusions

The optimization of the cmfs of individuals using a variation method has been described. In the variation method, metamer data has been employed as the source data.

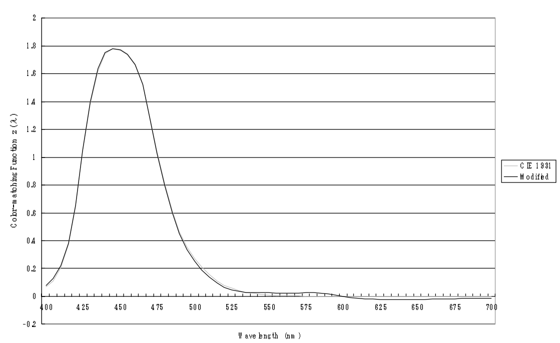
A feature of the optimization method is that the color difference in a metamer pair can be optimized to be 0 at a boundary condition in the variation method, and the smoothness of the modified cmf results from the cost function of the least mean square of modified cmf values in the variation method.



(a) modified $\bar{x}$



(b) modified $\bar{y}$

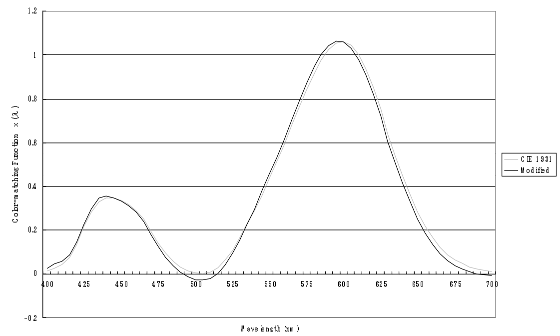


(c) modified $\bar{z}$

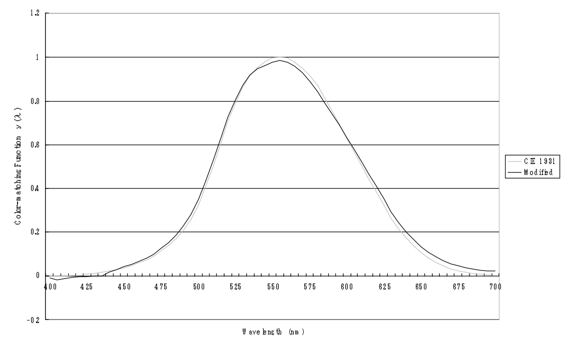
Figure 1 Modified cmfs for observer 1.

Experiments have derived the modified cmfs for individual observers and its validity in means of $\Delta E = 0$.

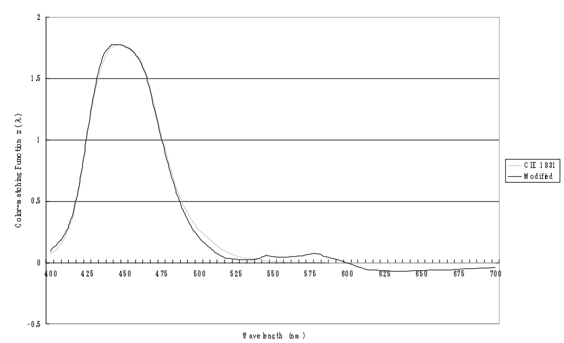
Hereafter, we will apply the optimization method to other problems in color science.



(a) modified $\bar{x}$



(b) modified $\bar{y}$



(c) modified $\bar{z}$

Figure 2 Modified cmfs for observer 2.

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Appendix

The relationship between the Hessian matrix of the cost function and the convexity of the cost function has been discussed in the text[4)6)]. The relationship between the Hessian matrix and the convexity of the truncated cost function employed in this paper has not been discussed. Theorem 1 and theorem 2 describe the relationship.

Theorems and a characteristic of the optimization method and the property of the solutions are also included.

[Definition]

A matrix M is defined to be semipositive if $0 \le q^t M q, \quad \forall q.$

[Theorem 1]

If the Hessian matrix of the cost function is semipositive at q $(q \in R)$, there exists a region of Δq values around q in which the truncated cost function $cf_t(q)$ is convex, where R indicates the defined range for the truncated cost function $cf_t(q)$.

[Proof]

Using a Taylor expansion based on the averaging theorem, the cost function cf is described as follows:

$$cf(q') = cf(q + \Delta q)$$
$$= cf(q) + \nabla cf(q)\Delta q + (1/2)(\Delta q)^t H(\tau q' + (1-\tau)q)\Delta q,$$
$$\exists \tau \ (0 < \tau < 1), \qquad (A.1)$$

where

$$\Delta q = q' - q,$$

τ : parameter of the averaging theorem.

On the other hand, the Taylor expansion with a high ordered term R_n is as follows:

$$cf(q') = cf(q) + \nabla cf(q)\Delta q + (1/2)(\Delta q)^t H(q)\Delta q + R_n \qquad (A.2)$$

The R_n term can be calculated using eqs.(A.1) and (A.2),

$$R_n = (\Delta q)^t (H(\tau q' + (1-\tau)q) - H(q))\Delta q / 2. \qquad (A.3)$$

The Hessian matrix of the cost function cf is semipositive at q and the following equation holds:

$$0 \le (1/2)(\Delta q)^t H(q)\Delta q. \qquad (A4)$$

The following relationship can be derived using eqs.(A.2) and (A.4),

$$cf(q') \ge cf(q) + \nabla cf(q)\Delta q + R_n. \qquad (A5)$$

For each value of $q' = q'_1,\ q' = q'_2,\ (q'_1 < q < q'_2)$,

$$cf(q'_1) \ge cf(q) + \nabla cf(q)\Delta q_1 + R_n^{(1)}, \qquad (A6)$$

$$cf(q'_2) \ge cf(q) + \nabla cf(q)\Delta q_2 + R_n^{(2)}, \qquad (A7)$$

where

$$\Delta q_1 = q'_1 - q,$$

$$\Delta q_2 = q'_2 - q,$$

The weighted combination of eqs.(A.6) and (A.7) is:

$$\omega cf(q'_1) + (1-\omega)cf(q'_2) \ge cf(q) + \nabla cf(q)(\omega q'_1 + (1-\omega)q'_2 - q) + (\omega R_n^{(1)} + (1-\omega)R_n^{(2)}) \qquad (A8)$$

By applying the relation $q = \omega q'_1 + (1-\omega)q'_2$ in eq.(A.8), the following equation is derived:

$$\omega cf(q'_1) + (1-\omega)cf(q'_2) \ge cf(\omega q'_1 + (1-\omega q'_2)) + (\omega R^{(1)} + (1-\omega)R^{(2)}) \qquad (A9)$$

and the values of $\Delta q_1, \Delta q_2$ values which satisfy the relation are found out:

$$|\,\omega R_n^{(1)} + (1-\omega)R_n^{(2)}\,| \le |\,\omega cf(q'_1) + (1-\omega)cf(q'_2) - cf(wq'_1 + (1-\omega)q'_2)\,|. \qquad (A10)$$

In eq.(A.10), the order of $\Delta q_1, \Delta q_2$ on the left side of eq.(A.10) is two orders higher than the order of $\Delta q_1, \Delta q_2$ on the right side of eq.(A.10). For sufficiently small $\Delta q_1, \Delta q_2$ values, eq.(A.10) is satisfied. Based on the relationship in eq.(A.10), $R_n^{(1)}, R_n^{(2)}$ terms can be ignored without changing the relationship in eq.(A.9), and the following relation holds:

$$\omega cf_t(q'_1) + (1-\omega)cf_t(q'_2) \ge cf(\omega q'_1 + (1-\omega)q'_2) . \qquad (A11)$$

Since $cf(q) = cf_t(q)$, $cf(\omega q'_1 + (1-\omega)q'_2) = cf_t(\omega q'_1 + (1-\omega)q'_2)$ holds. Therefore, the following relation is derived:

$$\omega cf_t(q'_1) + (1-\omega)cf_t(q'_2) \ge cf_t(\omega q' + (1-\omega)q'_2) . \qquad (A12)$$

The relation in eq.(A.12) is the condition of convexity for the function cf at q and various $\Delta q_1, \Delta q_2$ values construct a region around q in which the cost function is convex.

[Theorem 2]

If the truncated cost function $cf_t(q)$ is convex at a coordinate q, the Hessian matrix of the cost function is semipositive at the coordinate q.

[Proof]

Based on the assumption of the theorem and on the definition of cf_t,

$$\begin{aligned}\tau cf_t(q_1) + (1-\tau)cf_t(q_2) &\geq cf_t(\tau q_1 + (1-\tau)q_2)\\ &= cf_t(q_2 + \tau(q_1 - q_2))\\ &= cf(q_2) + \tau\nabla cf(q_2)(q_1 - q_2) + \tau(q_1 - q_2)H(q_2)\tau(q_1 - q_2)/2.\end{aligned} \quad (A.13)$$

By the modification of eq.(A.13), the following equation is derived,

$$\begin{aligned}\tau(cf_t(q_1) - cf_t(q_2)) &\geq \tau\nabla cf(q_2)(q_1 - q_2)\\ &+ (1/2)\tau(q_1 - q_2)H(q_2)\tau(q_1 - q_2),\end{aligned} \quad (A.14)$$

and divided by τ,

$$(cf_t(q_1) - cf_t(q_2)) \geq \nabla cf(q_2)(q_1 - q_2) + (1/2)(q_1 - q_2)H(q_2)\tau(q_1 - q_2), \quad (A.15)$$

Under the condition of $\tau \to 0$, the following equation is derived:

$$(cf_1(q_1) - cf_2(q_2)) \geq \nabla cf(q_2)(q_1 - q_2). \quad (A.16)$$

Let q_1, q_2 be $q_1 = q + \Delta q,\ q_2 = q$ and eq.(A.16) becomes:

$$cf_t(q + \Delta q) \geq cf_t(q) + \nabla cf(q)\Delta q, \quad (A.17)$$

and $cf_t(q) = cf(q)$, eq.(A.17) becomes as follows:

$$cf_t(q + \Delta q) \geq cf(q) + \nabla cf(q)\Delta q. \quad (A.18)$$

The Taylor expansion of the left side of eq.(A.18) is

$$cf_t(q + \Delta q) \geq cf(q) + \nabla cf(q)\Delta q + (1/2)(\Delta q)' H(q)\Delta q. \quad (A.19)$$

From eqs.(A.18) and (A.19), the following relationship is derived:

$$0 \leq (1/2)(\Delta q)' H(q)\Delta q, \quad (A.20)$$

and $H(q)$ becomes semipositive.

[Theorem 3]

The method derives the optimum solution based on the cost function CF (eq.(7)) restricted by constraints (eq.(8)), under the assumption that the continuity of the first-order derivative about cmfs is not considered.

[Proof]

Omitted.

[Theorem 4]

The solutions of the method are unbiased against the cmfs of an individual observer based on expectations.

[Proof]

Omitted.

[Characteristic]

The Hessian matrix of eq.(17) is semipositive for any value of q.

[Proof]

Omitted.