

Metamer Crossovers of Infinite Metamer Sets

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Abstract

For a fixed illuminant and observer there is a whole set of reflectances resulting in an identical response, these reflectances are called *metamers*. It can be shown analytically that all reflectances in each such set must intersect at least three times.

There is a large amount of literature arguing about the properties of these sets, in particular about the position and number of nodes of intersections. The results in the literature, based on relatively small data sets, vary in particular as a consequence of different methods used for generating metamers.

Using a new method based on statistical information from measured sets, metamer sets are generated. These *infinite metamer sets* are then studied for their inner structure in terms of cross-over behavior. The results presented here confirm the result of there being three major wavelengths of intersection. These are around 450nm, 540nm and 610nm.

1. Introduction

The human visual system is sensitive to the wavelength range of 400nm to 700nm. However, it is more or less sensitive to different wavelength regions within this range. Many studies have shown that there is particularly high sensitivity to small wavelength intervals centered around 450nm, 540nm and 610nm. More generally, these special or "prime" wavelength intervals are known to have a number of important properties:

Monitor Gamut Monitors equipped with primaries that peak sharply at around 450nm, 540nm and 610nm have been shown to have maximally large gamuts[3].

Sensor sharpening Linear combinations of the cone sensitivities that are "sharpest", peak at around 450nm, 540nm and 610nm. These sharp sensors have been shown to simplify colour–constancy and other image analysis tasks [3]. Moreover, the Bradford curves, used in CIE's colour appearance model CIECAM97s (which were designed to optimise the ability of the von Kries transformation to model illuminant change) peak around the same wavelengths.

Metamerism If a color input device's sensitivities are very narrow (in the limiting case have support only at a single wavelength) then it is easy to show that device metamers (spectral stimuli that induce the same device response) necessarily cross at the sensor wavelength positions. In terms of the position and frequency of crossover wavelengths of metamers for the human visual system, it is found that metamers tend to cross at around 450nm, 540nm and 610nm. We would expect this result if the human visual system sampled spectra in the same way as a narrow-band sensor device that only had sensitivities located at these wavelengths. Indeed, this is the case for a large corpus of real lights and real reflectances[9].

Printing If the spectral reflectances of printed inks have high reflectivity in the prime–wavelengths (and less outside these ranges) then illumination change for a print sample viewed under a pair of lights would adhere to the von Kries law (assuming sharp sensors anchored in the prime wavelength intervals). That is, choosing ink sets that enforce such reflectance profiles helps prevent metameric effects due to illuminant change. Most dye sets naturally enforce these kinds of reflectance profiles.

The size of the monitor gamut and the efficacy of spectral sharpened sensors to deliver von Kries adaptation are well established through rigorous mathematical proof. The debate about crossover wavelengths is in reality less conclusive. While the propensity of data over various studies indicate that crossover points tend to appear at the prime wavelengths, other authors have pointed out that metamers need not cross at these wavelengths [17]. One can synthesise metamers that only have crossover wavelengths at anti-prime wavelengths. That this is so is quite an important study. It is well known that the human eye can see thousands and thousands of distinct colour stimuli. Yet, traditional studies of metamers involve a small number of reflectances. Small enough that the loci of crossover wavelengths cannot, with any reasonable statistical significance, be thought to extrapolate to large data sets. On top of this problem is the fact that many antecedent studies are based on putatively "reasonable" metamers. In particular they are usually constructed by a linear combination of a smooth metamer basis. While it is true that most real reflectances are smooth, there is no guarantee that a mathematical construction of reflectances, that implements this intuition, has any basis in nature.

In this paper we tackle both these problems head on. First, we look at the statistics of real reflectances and consider the circumstances through which nature might generate metamers. In particular a collage of n reflectances viewed from a far enough distance generates a single new aggregate reflectance. By averaging in this way (in fact by taking convex combinations of real reflectances) we can generate an infinite set of real reflectances. Many of these reflectances will induce the same device response under given lighting conditions, so the collage idea allows us to generate infinite metamer sets. These infinite sets are predicated only on the statistics of real natural reflectances. At a second stage we introduce methods for quantifying the cardinality of the set of infinite metamers which cross over at particular wavelengths.

Our analysis is performed for a number of illuminants and reflectance data sets. In all cases we observe that there are three distinct places in the visible spectrum where the infinite, statistically plausible metamers tend to cross – crossovers statistically are found only around the prime wavelengths: 450nm, 540nm and 610nm. Our results serve to strengthen previous studies on metamer crossovers and more generally lend support for the theory that prime wavelengths are fundamental to the understanding of colour vision and colour imaging.

In the following section previous work in the area of metamer crossovers is reviewed and limitations and problems with the approaches are pointed out. Section 3 presents the new infinite metamer approach. Experimental results are discussed in section 4.

2. Metamer Crossovers

Two spectrally different reflectances resulting in an identical response under a given illuminant are *metamers*. The necessarily convex set of all such reflectances is called the *metamer set*.

Early work in the study of metamers, has showed that for two reflectances to be metameric to the human visual system, they need to cross at least three times across the visible spectrum [19]. The position of these crossovers has been the source of much subsequent discussion and debate.

For smooth metamer pairs, Thornton [20] found that there are three statistically significant intervals of crossovers: 448nm $\pm$ 4nm, 537nm $\pm$ 3nm and 612nm $\pm$ 8nm. He concluded that these three narrow band wavelength ranges correspond to the human visual system's peak sensitivity areas and he labeled them *prime wavelengths*.

Ohta and Wyszecki [17] questioned the necessity of these three particular wavelength ranges, and using numerical methods they produced two sets of pairs of metamers which reproduced Thornton's findings to a lesser degree.

Other works [2, 16, 1] studied the crossover problem from different perspectives (crossovers for a varying metamerism index, crossover behavior of the statistically most important illuminant, etc.), and concluded that for natural reflectances it is indeed the case that they cross at the prime wavelengths.

At first glance then, taken together, these studies apparently suggest that metamers should cross over around the prime wavelengths. Unfortunately, few of the above studies are statistically significant enough to really speak about general properties of metamers. First, previous studies are based only on a small number of metamers. Moreover, for a given color response (XYZ) usually only a small number of metamers are considered (in fact in most studies a single pair). Perhaps more serious than these problems is the fact that metamers were generated synthetically and so need say nothing about typical metamers encountered in the real world.

Some of these concerns were addressed in a recent work by the authors [11] that developed a numerical procedure for generating large numbers of statistically plausible metamers for a given device response.

2.1. Generating Natural Metamers

Let us assume that continuous spectra might be represented by sample values at a discrete number of sampling wavelengths (in fact by values at 400nm through 700nm at a 10nm sampling distance). It follows that 31-vectors $\mathbf{r}$ and $\mathbf{e}$ can be used to represent surface spectral reflectance and illumination spectral power distribution. The 31×3 matrix $\overline{\mathbf{X}}$ denotes the three CIE XYZ color matching functions (one function per column). Now, for flat, Lambertian surfaces illuminated by a diffuse illuminant of known spectral power distribution, the CIE tristimulus values can be calculated using the following matrix equation:

$$\mathbf{r^T} D(\mathbf{e})\overline{\mathbf{X}} = \mathbf{x^T} \quad (1)$$

where $D()$ is an operator mapping elements of a vector into the diagonal elements of a matrix, and ^{T} is the transpose operator.

A given $\mathbf{x}$ may be induced by many $\mathbf{r}$-s. If we think of $\mathbf{r}$ as an unknown to be solved for in eq. (1) we come to understand this observation in more detail. The right-hand side of (1) contains 3 knowns but we are trying to solve for the 31 unknowns on the left-hand side. Clearly there is no unique solution to this problem. In fact Cohen and Kappauf have shown that there is a 28 dimensional space of solutions [5, 6, 7].

However, in practice real reflectances inhabit only a small part of 31-dimensional vector space. Indeed, using techniques such as characteristic vector analysis it is possible to find a small number of basis vectors (between 3 and 6), which suffice to model variation in measured reflectance data [18, 8, 13, 15]. Not only do such analyses help us represent reflectances more concisely but it is of considerable import when we come to look at metamerism.

Let us suppose we have m basis vectors $\mathbf{b}$ collected in a matrix $\mathbf{B}$. Let $\mathbf{r} = \sigma_1\mathbf{b_1} + \sigma_2\mathbf{b_2} + \ldots + \sigma_\mathbf{m}\mathbf{b_m}$ (reflectance is uniquely defined by the σ weights). Now we can rewrite eq. (1) by substituting $\mathbf{r} = \mathbf{B}\sigma$:

$$\sigma^T\mathbf{B^T}D(\mathbf{e})\overline{\mathbf{X}} = \mathbf{x^T} \qquad (2)$$

Equation (2) represents 3 equations of m unknowns (the σ vector). It is easy to show that the set of solutions to eq. (2) is $m-3$ dimensional. Notice if $m = 3$ then the solution is 0 dimensional: i.e. there is a single point solution [14]. Indeed, we expect this to be the case since $\mathbf{B^T}D(\mathbf{e})\overline{\mathbf{X}}$ is a 3×3 matrix which has a unique inverse.

Let us now consider how eq. (2) might be solved for the case of $m > 3$. This set of linear equations can now be solved by decomposing the solution into a sum of two partial solutions [12]. Without loss of generality let us assume that the first 3 rows of $\mathbf{B^T}D(\mathbf{e})\overline{\mathbf{X}}$ are non zero and the last (m-3) rows equal zero. That is the human visual system, given illuminant $\mathbf{e}$, can only "see" the first three basis functions – the human visual system is orthogonal to the last $m-3$ basis functions. Now it is easy to find a particular solution: we simply find the linear combination of $\mathbf{b_1}$, $\mathbf{b_2}$ and $\mathbf{b_3}$ that induces the required tristimulus. The second part is a solution involving the remaining $m-3$ basis functions. By definition, the visual response to $\mathbf{b_i}$ ($i > 3$) is zero. Each such $\mathbf{b_i}$ is called a metameric black. It is also apparent that if $\mathbf{b_i}$ and $\mathbf{b_j}$ induce a zero response then so does $\alpha\mathbf{b_i} + \beta\mathbf{b_j}$ (visual response is linear). It follows that the general set of all metamers have the form: particular solution plus any linear combination of metameric blacks [10].

Of course the fact that we might add metameric blacks together and then add this sum to the particular solution does not mean that we end up with plausible reflectances. Clearly, in forming metamers there are constraints that we must enforce. Reflectances are non-negative (no less than no light is reflected) and less than or equal to one (no more than all light is reflected). There are further constraints that apply to the individual basis functions. These can be determined experimentally i.e. for a corpus of real reflectances we determine the range within which the factors applied to the basis functions must fall.

Both the physical (bigger than or equal to 0% less than or equal to 100%) and statistical weight constraints are linear inequalities. It follows that for a given tristimulus value (for which we wish to generate metamers) we can use linear programming [4] to find a region of weight space, actually a hyper-rectangle, in which metamers must lie. By sampling this hyper-rectangle we can generate an arbitrarily large number of metamers.

In a study [11] we took a number of reflectance sets, projected them to XYZ tristimulus values, and using a basis derived from the same reflectance set, we recovered metamer set approximations using the above method. For each set we then examined the crossover behavior of the samples within the set, and we arrived again at three prime peaks around 450nm, 540nm and 610nm – the prime wavelengths. This approach represented a significant improvement over previous studies. A large number (thousands) of metamers were generated and all of these were physically plausible. However, the approach was not perfect. The fact that metamers were generated through sampling was unsatisfactory. How many samples are necessary? Did the sampling strategy bias the crossover results to favor one wavelength over another?

In order to address these issues we set out to solve for the analytic solution of the number of crossover metamers as a function of wavelength.

3. Infinite Natural Metamer Sets

Linear inequalities, which define our metamer set, can be viewed as a set of half-spaces (they are linear divides that split reflectance space into two parts). Taken together a set of linear inequalities describes a volume (a convex region of reflectance space). Given linear inequalities describing the set of all real reflectances (this defines a convex set in weight space), and the set of all signals inducing a given XYZ tristimuli (an affine plane in weight space) we can solve for the metamer set by intersecting these two geometric bodies. The result is a closed convex region of weight space. Relative to this metamer set we wish to count the number of metamers that crossover at a given wavelength. The notion of crossovers can also be expressed as linear inequalities (and so by hyper-planes). However, care must be taken here. The hyper-plane that describes a crossover point for an increasing function is different from the hyper-plane for a decreasing function. Each hyper-plane must be intersected with the metamer set separately. The sum of volumes for the two intersections returns a volume proportional to the cardinality of the metamer set that has a particular crossover wavelength.

As we are interested in the histogram of metamer intersections, we go sequentially through all wavelengths of the visible range and find the corresponding sub-set of the metamer set which represents metamers crossing at the particular wavelength. This approach gives us the freedom to get precise results, to use a sampling of the visible range as fine as we need, to avoid sampling the convex hull of weights of the metamer set and to arrive at an infinite set of metamers. It is also much faster to arrive at this analytic result than to run the sampling algorithm.

Those readers familiar with notions of convexity, hyperplanes and geometric calculations – convex set intersection and convex hull computation – will realise that the program for determining the cardinality of infinite metamer sets (of reflectances with the same crossover point) is quite non-trivial. The main bulk of our code is written in around 120 lines in Matlab and we make external calls to the Qhull algorithm (available at www.geom.umn.edu/software/qhull/). The interested reader is encouraged to contact us for more details.

4. Results

A number of reflectance sets were taken as a basis for the experiments. These were a set of 462 Munsell colour chips [18], a set of 170 object reflectances[21] and a set of 120 DuPont colour chips. The convex hull of the tristimuli of each of these sets is calculated and consequently sampled using a uniform, tetrahedral, sampling so as to arrive at approximately 400 points inside the hull.

For each tristimulus value the convex hull of it's metamer set is calculated and intersected sequentially by the half-spaces defining a crossover at each wavelength in the visible range, at a 10nm sampling. The intersection's volume is calculated and used as a representation of the fraction of the convex hull which corresponds to metamers intersecting at a particular wavelength. The final histogram is then the sum of all histograms of the interior points of the convex set of tristimuli.

Each experiment was carried out for illuminants D65, A and TL84 (fluorescent) and 6 basis vectors were used to represent the reflectances, as this basis covered over 99% of variance in each studied set.

The results are strikingly clear, and for all illuminants and data sets three distinct peaks are clearly present, all of which are around the prime wavelengths 450nm, 540nm and 610nm.

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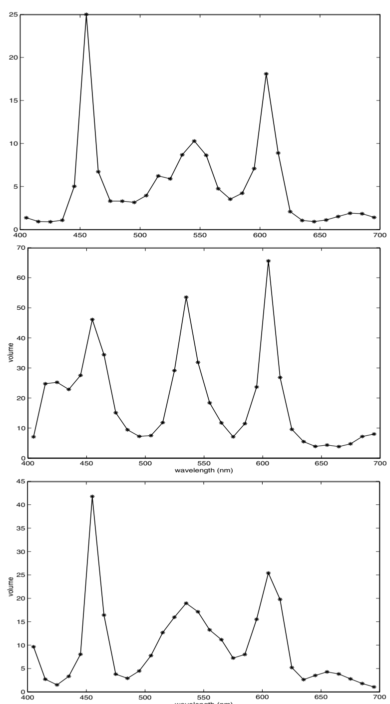

Figure 1: Metamer crossover histograms for illuminant D65 (from top to bottom: the Munsell data set, the Object data set and the DuPont data set).

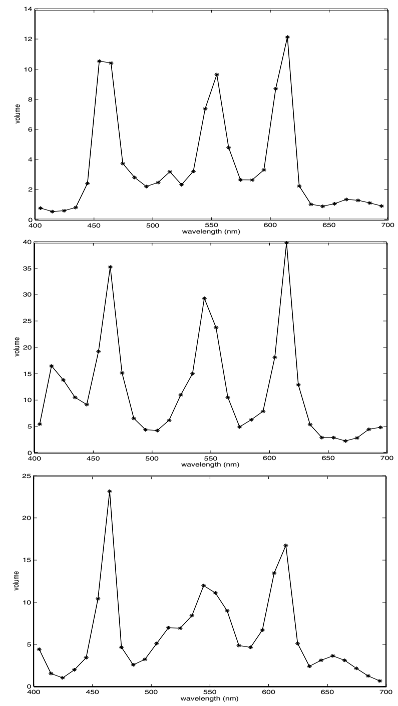

Figure 2: Metamer crossover histograms for illuminant A (from top to bottom: the Munsell data set, the Object data set and the DuPont data set).

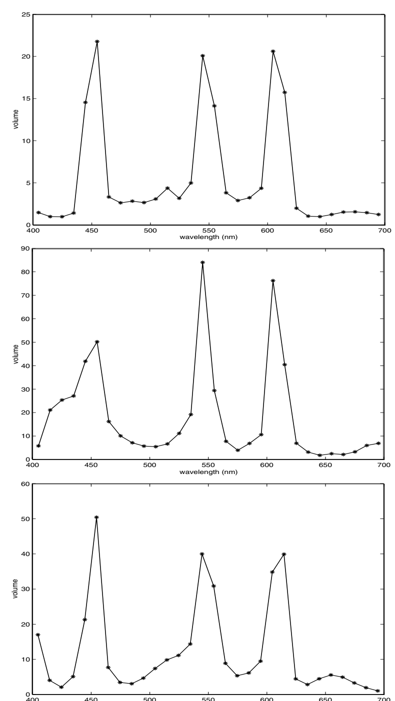

Figure 3: Metamer crossover histograms for illuminant TL84 (from top to bottom: the Munsell data set, the Object data set and the DuPont data set).