White-Point Preservation Enforces Positivity

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Abstract

It is commonplace to use a 3×3 linear transform to map device RGBs to XYZs. Two particular types of transforms have been developed based on the assumptions that we either maximally ignorant or maximally prescient about the world. Under the maximum ignorance assumption, it is assumed that nothing is known about the spectral statistics of the world and so the best correction transform is the one that maps device spectral sensitivities so they are as close to observer sensitivities as possible. Under maximum prescience, we know the spectral statistics that we will observe and so the maximally prescient transform maps, with minimum error, the RGBs (that we know we will see) onto corresponding XYZs. In general the two assumptions lead to quite different color corrections.

In previous work we have argued against total ignorance or prescience and have instead developed compromise transforms. Our work is based on two observations. First, one is never completely ignorant about the world—color signal spectral power distributions are everywhere all positive. Second, it is accepted that it is much more important to correct some colors than other. In particular, white is central to color vision and color imaging, so it is imperative that white should always look right.

However, to date these two compromise solutions have been studied in isolation. Surely, it would be advantageous to combine the constraints of whiteness and positivity? In fact we show that this is not the case: by preserving white we enforce positivity. This is an important result. Not only does it add to our understanding of color correction, but it helps explain color correction results published in the literature (the assumptions of positivity and white-preservation lead to very similar results). Moreover, it helps us to derive a new measure for assessing the goodness (color correctability) of camera sensors that is strictly less pessimistic (and more accurate) than the existing Vora Value.

1. Introduction

Color correction, mapping device RGBs to XYZs, is a central problem in color imaging[1]. After many years of research, standardized correction methods are evolving. In the domain of digital cameras, the standard method involves transforming RGBs to XYZs using the ‘appropriate’ linear transform. The definition of ‘appropriate’ depends on the assumptions that are made when deriving the transform. Recently, two weak, and so easily justifiable, assumptions have emerged. First, reflectances must be positive[2] and second, in carrying out color correction white should be preserved[3, 4]. Both assumptions have been to shown to deliver favourable color correction[4, 2]. Somewhat surprisingly, the constraints of white preservation and positivity have, to date, been examined in isolation. In this paper we study both constraints together.

We begin by re-examining the white-point preserving color correction transform. While the idea of preserving white is a simple one—all we want to do is to get white looking right—expressing it and manipulating it mathematically, is somewhat non-trivial[4]. Indeed, the original mathematical argument is quite complex. Thus, in the first part of this paper we spend considerable effort boiling down the white-preservation argument to a simpler set of equations; an equation set that is easy to implement[1].

At a second stage we adopt the maximum ignorance assumption: the idea that all spectra with both positive and negative power occur with equal likelihood. It is well known that the MI assumption often simplifies mathematical argument[5] and it does so again here. Indeed, we show that the white-point preserving transform depends on three ’projectors’[6, 7]: the projector for the camera, the projector for white and the projector for the space that is orthogonal to the camera. In contrast, conventional least-squares involves only the projector for the camera. It is not too important that the reader understands projectors (hopefully,

[1] Please look at http://color.derby.ac.uk/ graham/ for an implementation in the S-Plus statistical programming language

those that do will appreciate the elegance of the result). What is important is that the derived expression is again very simple to code and implement[1]; in fact, it is a single closed form equation.

The MI assumption, useful as it is in simplifying mathematical argument, is of questionable value. In particular, the MI assumption accounts for spectra with positive and negative power. Yet, there is no physical basis for negative power. This discrepancy led Finlayson and Drew to develop the maximum ignorance with positivity assumption (MIP)[2]: the assumption that nothing is known about the color signals except that they are physically realizeable (everywhere positive).

Preserving white under the conventional MI conditions improves color correction[3]. So too, does imposing positivity when white is not explicitly preserved[2]. One might expect that a combined correction transform would deliver better correction still. This is in fact not the case. The central result of this paper is to prove that the white-point preserving transforms derived under MI conditions, with and without positivity, are identical. *Preserving white enforces positivity*.

This result helps to clarify experimental results that are reported in the literature. Hubel et al. have shown that a white-point preserving transform derived under MI conditions without positivity, WPPMI, delivers excellent correction performance[8]. That this is so is not surprising since we now know that Hubel et al. were, albeit implicitly, assuming only positive spectra. This result adds further weight to the proposal, currently under consideration by the ISO and IEC technical committees, that the white-point preserving correction transform derived under MI conditions be made a standard[9].

In the final part of this paper we consider how well we can expect the WPPMI transform to work given particular camera spectral sensitivities. We answer this question by extending the Vora Value a standard measure of filter goodness[5], to the white preserving case. Here, there **is** a difference between the assumptions of MI and MIP. The Vora Value with positivity, is always substantially greater than that calculated under MI conditions. It follows that the new measure is more accurate since it is based only on legitimate assumptions about physical spectra

In section 2 of this paper we introduce the white point preserving color correction transform and examine its operation under MI assumptions. In section 3, we show that preserving white, under the MI assumptions, also enforces positivity. Based on the theoretical development, a new measure of the colorimetric performance of a color filter set (for scanners and cameras) is presented in section 4. The paper concludes in section 5.

2. White-point preserving color correction

Let $\mathcal{N}$ be a $n \times 3$ matrix of camera RGBs and $\mathcal{V}$ a corresponding matrix of XYZs for n color patches. To ease the mathematics that follows we will assume that the XYZs are white balanced. That is, if (x, y, z) is the color coordinate induced by a surface and (x_w, y_w, z_w) is the tristimulus for white then (x, y, z) is set to $(x/x_w, y/y_w, z/z_w)$; white equals $(1, 1, 1)$. The corresponding camera white RGB is denoted by the vector $\underline{w}$

The aim of color correction is to find linear combinations of the columns of $\mathcal{N}$ that are as close as possible to the columns of $\mathcal{V}$. To see how this might be done let $\underline{v}$ and $\underline{c}$ denote an $n \times 1$ target vector (one column of $\mathcal{V}$) and a 3×1 coefficient vector (the linear combination to be solved for) respectively. In standard linear regression, we solve for the coefficient vector $\underline{c}$ that minimizes:

$$I = |\mathcal{N}\underline{c} - \underline{v}| \tag{1}$$

where $|.|$ is the L_2 norm (vector length or Root Mean Square Error). Using the fact that $|\underline{a}| = \sqrt{\underline{a}.\underline{a}}$ (the magnitude of $\underline{a}$ is the square root of the dot product of $\underline{a}$ with itself), I^2 can be written as:

$$J = I^2 = \underline{c}^t\mathcal{N}^t\mathcal{N}\underline{c} - 2\underline{c}^t\mathcal{N}^t\underline{v} + \underline{v}^t\underline{v} \tag{2}$$

The $\underline{c}$ which minimizes (2) is found by differentiating J and equating to the zero vector $\underline{0}$:

$$\frac{\delta J}{\delta \underline{c}} = 2\mathcal{N}^t\mathcal{N}\underline{c} - 2\mathcal{N}^t\underline{v} = \underline{0} \tag{3}$$

It follows that

$$\underline{c} = [\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t\underline{v} \tag{4}$$

Equation (4) is the solution to the least-squares (LS) color correction problem. We can in fact solve for all three coefficient vectors, that is solve for the X, Y and Z coefficient mappings together:

$$\mathcal{C} = [\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t\mathcal{V} \tag{4a}$$

$\mathcal{C}$ is the 3×3 matrix that best maps RGBs to XYZs; that is, $\mathcal{C}$ minimizes $|\mathcal{V} - \mathcal{N}\mathcal{C}|$.

In white-point preserving color correction we wish to minimize (1) with the additional constraint that whites are corrected without error. To see how this is done let us add a Lagrange constraint term to (2):

$$J = \underline{c}^t\mathcal{N}^t\mathcal{N}\underline{c} - 2\underline{c}^t\mathcal{N}^t\underline{v} + \underline{v}^t\underline{v} + \lambda(\underline{c}^t\underline{w} - 1) \tag{5}$$

Differentiating with respect to the Lagrange multiplier λ and equating to $\underline{0}$ we have:

$$\frac{\partial J}{\partial \lambda} \;=\; \underline{c}^t\underline{u} - 1 = 0 \;\Rightarrow\; \underline{c}^t\underline{w} = 1 \tag{6}$$

Relation (6) tells us that when we find the stationary point of J we must have $\underline{c}^t\underline{w} = 1$. This is precisely the condition that we need for the white-point preserving minimization (remember white for the standard observer is (1,1,1) and white for the camera is $\underline{w}$). Differentiating (5) with respect to $\underline{c}$ and equating to $\underline{0}$:

$$\frac{\partial J}{\partial \underline{c}} \;=\; 2\mathcal{N}^t\mathcal{N}\underline{c} - 2\mathcal{N}^t\underline{v} + \lambda\underline{w} \;=\; \underline{0} \tag{7}$$

Taking (6) and (7) together and applying some algebraic manipulation it can be shown that:

$$\underline{c} \;=\; [\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t\underline{v} + [\mathcal{N}^t\mathcal{N}]^{-1}\underline{w}\frac{(1 - \underline{w}^t[\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t\underline{v})}{(\underline{w}^t[\mathcal{N}^t\mathcal{N}]^{-1}\underline{w})} \tag{8}$$

It is useful to look at the application of $\underline{c}$ to the matrix of camera responses $\mathcal{N}$ because in doing so we uncover some useful mathematical structure:

$$\mathcal{N}\underline{c} \;=\; \mathcal{N}[\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t\underline{v} + \mathcal{N}[\mathcal{N}^t\mathcal{N}]^{-1}\underline{w}\frac{(1 - \underline{w}^t[\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t\underline{v})}{(\underline{w}^t[\mathcal{N}^t\mathcal{N}]^{-1}\underline{w})} \tag{8a}$$

The matrix $\mathcal{N}[\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t$ is the projector for the space spanned by the columns of $\mathcal{N}$[10]. The projector of $\mathcal{N}$, denoted $P(\mathcal{N})$, has a number of attractive properties not least of which is the property that if $\underline{x}$ is an arbitrary n-dimensional vector then $P(\mathcal{N})\underline{x}$ is the closest vector in the space spanned by the columns of $\mathcal{N}$ to $\underline{x}$. Other properties of projectors that we exploit include: $\mathcal{P}^t = \mathcal{P}$ (symmetry) and $\mathcal{P}\mathcal{P} = \mathcal{P}$ (idempotency).

Let us choose an n-vector $\underline{W}$ such that $\mathcal{N}[\mathcal{N}^t\mathcal{N}]^{-1}\underline{w} = P(\mathcal{N})\underline{W}$ or in other words $\mathcal{N}^t\underline{W} = \underline{w}$ (it is always possible to make this substitution). The importance of this step will become clear when we come to examine the maximum ignorance case. Using this result we rewrite (8a):

$$\mathcal{N}\underline{c} \;=\; P(\mathcal{N})\left(\underline{v} + \underline{W}\frac{1 - \underline{W}^tP(\mathcal{N})\underline{v}}{\underline{W}^tP(\mathcal{N})\underline{W}}\right) \tag{8b}$$

The white-point preserving transform taking RGBs to Xs (Ys or Zs) is a simple projection of Xs onto the space spanned by the camera measurements $\mathcal{N}$ (this is simply conventional least squares) plus the projection of some offset term.

Before we can simplify (8b) further we need to consider how light and sensor interact in forming a device response. In equation (9) $F(\lambda)$ denotes the spectral sensitivity of a device and $S(\lambda)$ a spectrum of light. The integral over the visible spectrum ω defines sensor response:

$$f \;=\; \int_\omega F(\lambda)S(\lambda)d\lambda \tag{9}$$

where $F(\lambda)$ represents one of the XYZ color matching functions or equally one of the RGB camera color channels. If we represent the continuous functions $F(\lambda)$ and $S(\lambda)$ by vectors $\underline{F}$ and $\underline{S}$ of values at m discrete sample points, then (9) can be rewritten as:

$$f \;=\; \int_\omega F(\lambda)S(\lambda)d\lambda \;\approx\; \sum_{i=1}^{m} F_iS_i \;=\; \underline{F}\cdot\underline{S} \tag{10}$$

Let the $m \times 3$ matrices $\mathcal{X}$ and $\mathcal{R}$ denote color matching functions and camera sensitivities respectively. By stacking the n spectra of light in the n rows of the $n \times m$ matrix $\mathcal{S}$:

$$\mathcal{N} \;=\; \mathcal{S}\mathcal{R} \quad , \quad \mathcal{V} \;=\; \mathcal{S}\mathcal{X} \tag{11}$$

Denoting a single column of $\mathcal{X}$ as $\underline{x}$ and substituting (11) into (8b):

$$\mathcal{S}\mathcal{R}\underline{c} \;=\; P(\mathcal{S}\mathcal{R})\left(\mathcal{S}\underline{x} + \underline{W}\frac{(1 - \underline{W}^tP(\mathcal{S}\mathcal{R})\mathcal{S}\underline{x})}{(\underline{W}^tP(\mathcal{S}\mathcal{R})\underline{W})}\right) \tag{12}$$

Let us look again at Equation (8a) and the role that the color signal matrix $\mathcal{S}$ plays in the regression. Again, substituting $\mathcal{S}\mathcal{R}$ and $\mathcal{S}\mathcal{X}$ for $\mathcal{N}$ and $\mathcal{V}$, we can see that the following relations hold:

$$\begin{aligned} [\mathcal{N}^t\mathcal{N}]^{-1}\mathcal{N}^t\underline{v} \;&=\; [\mathcal{R}^t\mathcal{S}^t\mathcal{S}\mathcal{R}]^{-1}\mathcal{R}^t\mathcal{S}^t\mathcal{S}\underline{x} \\ [\mathcal{N}^t\mathcal{N}]^{-1} \;&=\; [\mathcal{R}^t\mathcal{S}^t\mathcal{S}\mathcal{R}]^{-1} \end{aligned} \tag{13}$$

It follows that the white-point preserving fit depends only on the spectral sensitivities of the camera, the spectral sensitivities of the standard observer and the $m \times m$ spectral correlation matrix $\mathcal{S}^t\mathcal{S}$.

Under the conventional MI assumption, all spectra with bounded power, say in the interval $[-1, 1]$ (at each wavelength), are deemed equally likely. It can be shown that the corresponding spectral correlation matrix is equal to:

$$\mathcal{S}^t\mathcal{S} = 0.5\mathcal{I} \tag{14 MI}$$

In considering the white-preserving regression, that we can set $\mathcal{S} = \mathcal{I}$ since in so doing we recreate the required MI spectral correlation matrix: $\mathcal{I}^t\mathcal{I} = \mathcal{I}$ (note the scalar 0.5 in (14) is not important in the following argument). Substituting $P(\mathcal{I}\mathcal{R})$ for $P(\mathcal{N})$ in (8b):

$$\mathcal{R}\underline{c} \;=\; P(\mathcal{R})\left(\underline{x} + \underline{W}\frac{(1 - \underline{W}^tP(\mathcal{R})\underline{x})}{(\underline{W}^tP(\mathcal{R})\underline{W})}\right) \tag{15}$$

In (15) the vector $\underline{W}$ plays the same role as before though here it is an m-vector which, when projected onto the camera spectral sensitivities equals the camera response for white. Let us now assume that the camera is balanced for white:

$$\mathcal{X}^t\underline{U} \ = \ \mathcal{R}^t\underline{U} \ = \ (1,1,1)^t \tag{16}$$

where $\underline{U}$ denotes the spectrum of a perfect white diffuser (an m-vector that is all 1s). Clearly, $\underline{W}$ must equal $\underline{U}$. Rewriting (15):

$$\mathcal{R}\underline{c} \ = \ P(\mathcal{R})\left(\underline{x} + \underline{U}\frac{\underline{U}^t\underline{x} - \underline{U}^tP(\mathcal{R})\underline{x}}{(\underline{U}^tP(\mathcal{R})\underline{U})}\right) \tag{17}$$

and this simplifies to:

$$\mathcal{R}\underline{c} \ = \ P(\mathcal{R})\left(\mathcal{I} + \frac{\underline{U}\underline{U}^t[\mathcal{I} - P(\mathcal{R})]}{(\underline{U}^tP(\mathcal{R})\underline{U})}\right)\underline{x} \tag{18}$$

In this form, the spectral sensitivities of the color matching functions occur only once, and so the R, G and B camera sensitivities can be considered together:

$$\mathcal{R}\mathcal{C} \ = \ P(\mathcal{R})\left(\mathcal{I} + \frac{\underline{U}\underline{U}^t[\mathcal{I} - P(\mathcal{R})]}{(\underline{U}^tP(\mathcal{R})\underline{U})}\right)\mathcal{X} \tag{19}$$

Equation (19) is really remarkably simple and quite elegant. In the parlance of projectors, $P(\mathcal{R})$ is a projector for camera spectral sensitivities, $\mathcal{I} - P(\mathcal{R})$ is the projector for the space orthogonal to the camera (the spectra the camera cannot see) and $\underline{U}\underline{U}^t$ is proportional to the projector for white. In contrast, simple least-squares regression under MI conditions depends only on the $P(\mathcal{R})$. While (19) is more complex than conventional linear regression we believe it to be simpler, and more intuitive than higher order polynomial regression.

The correction transform $\mathcal{C}$ is equal to:

$$\mathcal{C} \ = \ \mathcal{R}^+\left(\mathcal{I} + \frac{\underline{U}\underline{U}^t[\mathcal{I} - P(\mathcal{R})]}{(\underline{U}^tP(\mathcal{R})\underline{U})}\right)\mathcal{X} \tag{20}$$

where $\mathcal{R}^+ = [\mathcal{R}^t\mathcal{R}]^{-1}\mathcal{R}^t$. To convince yourself of this, note that $\mathcal{R}\mathcal{C}$ returns the expression in (19).

3. Preserving white with positivity

The maximum ignorance assumption is flawed because it allows spectra to have negative power and this is physically impossible. To remedy this problem the maximum ignorance with positivity (MIP) assumption was proposed[2]: all all-positive spectra are equally likely. Under MIP conditions, spectra of light have power at each wavelength chosen uniformly randomly in the interval $[0,1]$. The corresponding $m \times m$ spectral correlation matrix equals[2]:

$$\mathcal{S}^t\mathcal{S} = (1/4)\underline{U}\underline{U}^t + (1/12)\mathcal{I} \qquad (21 \qquad \text{MIP})$$

To ease notation, WPP($\mathcal{R},\mathcal{X},\mathcal{S}^t\mathcal{S}$) denotes the white-point preserving transform that takes camera $\mathcal{R}$ to observer $\mathcal{X}$ for the spectral correlation $\mathcal{S}^t\mathcal{S}$ subject to the constraint that white is preserved.

Theorem: The white point preserving transform derived under the maximum ignorance with positivity is exactly the same as the white point preserving transform derived under MI (without positivity) conditions.

$$\text{WPP}(\mathcal{R},\mathcal{X},0.5\mathcal{I}) \ = \ \text{WPP}(\mathcal{R},\mathcal{X},(1/4)\underline{U}\underline{U}^t + (1/12)\mathcal{I})$$

Proof Sketch:

1. By definition a white-point preserving transform maps white correctly
2. Any light spectrum $\underline{S}$ can be written as a linear combination of the perfect white diffuser $\underline{U}$ and a component orthogonal to white $\underline{O}$:

$$\underline{S} \ = \ \alpha\underline{U} + \underline{O}, \quad (\underline{O}\cdot\underline{U} = 0)$$

3. Let $\mathcal{P}$ be a linear projector that takes the set of spectra stacked in the rows of $\mathcal{S}$ and returns the part of each spectrum orthogonal to $\underline{U}$. Projector $\mathcal{P}$ minimizes

$$|\mathcal{S}\mathcal{P} - S|, \quad (\mathcal{S}\mathcal{P}\underline{U} = \underline{0})$$

4. After projection by $\mathcal{P}$ the spectral correlation matrix $\mathcal{S}^t\mathcal{S}$ is equal to $\mathcal{P}\mathcal{S}^t\mathcal{S}\mathcal{P}$ (because $\mathcal{P}$ is a projector it is a symmetric matrix).
5.

$$\text{WPP}(\mathcal{R},\mathcal{X},\mathcal{S}^t\mathcal{S}) = \text{WPP}(\mathcal{R},\mathcal{X},\mathcal{P}\mathcal{S}^t\mathcal{S}\mathcal{P}).$$

(by definition white is corrected without error)

6.

$$\mathcal{P}^t[(1/4)\underline{U}\underline{U}^t + (1/12)\mathcal{I}]\mathcal{P} \ = \ (1/12)\mathcal{P}\mathcal{P} = (1/12)P.$$

7.

$$\text{WPP}(\alpha\mathcal{R},\alpha\mathcal{X},\mathcal{S}^t\mathcal{S}) = \text{WPP}(\mathcal{R},\mathcal{X},\alpha^2\mathcal{S}^t\mathcal{S}) = \text{WPP}(\mathcal{R},\mathcal{X},\mathcal{S}^t\mathcal{S})$$

8. Combining 5, 6 and 7, it follows that

$$\text{WPP}(\alpha\mathcal{R}, \alpha\mathcal{X}, 0.5\mathcal{I}) =$$

$$\text{WPP}(\alpha\mathcal{R}, \alpha\mathcal{X}, (1/12)\mathcal{I}) =$$

$$\text{WPP}(\mathcal{R}, \mathcal{X}, [(1/4)\underline{U}\underline{U}^t + (1/12)\mathcal{I}])$$

4. Preserving white, positivity and filter goodness

The Vora Value[5] measure of filter goodness quantifies the error between XYZs and corrected RGBs. If camera RGBs, are stored in the $n \times 3$ matrix $\mathcal{N}$ and XYZs in a matrix $\mathcal{V}$ and the 3×3 correction transform is denoted $\mathcal{C}$, the Vora Value equals:

$$\text{VoraValue} \;=\; 1 - \frac{|\mathcal{V} - \mathcal{N}\mathcal{C}|^2}{|\mathcal{V}|^2} \qquad (22)$$

As the Vora Value becomes closer to one, so the camera becomes more colorimetric; that is, the camera samples light more like the standard observer. However, one finds that most devices yield Vora Values very close to one (devices almost always have Vora Values > 0.9) so it is interesting to look at one minus the Vora Value since these values have greater, device dependent variation (all the variance is in the difference between 1 and 0.9). The Vora Error measure is equal to:

$$\text{VoraError} \;=\; \frac{|\mathcal{V} - \mathcal{N}\mathcal{C}|^2}{|\mathcal{V}|^2} \qquad (23)$$

Equation (23) is simply the sum of squares differences between XYZs and corrected RGBs normalized by the sum of squared XYZs. In fact Vora and Trussell's measure is sometimes more particular than equation (23). Rather than using XYZ responses themselves, decorrelated counterparts are often used (where responses are decorrelated through the application of a linear transform). In the text that follows we will keep XYZs. However, in so doing we do not lose any generality in our argument (we could equally well use any basis with the proviso that it is first normalized to white).

The white-point preserving Vora Error (WPPVE) is defined as:

$$\text{WPPVE}; \;=\; \frac{|\mathcal{S}\mathcal{X} - \mathcal{S}\mathcal{R}\text{WPP}(\mathcal{R}, \mathcal{X}, \mathcal{S}^t\mathcal{S})|^2}{|\mathcal{S}\mathcal{X}|^2} \qquad (24)$$

where $\mathcal{X}$ are the observer curves, $\mathcal{R}$ camera spectral sensitivities $\mathcal{S}$ color signal spectra. Henceforth we substitute $\mathcal{C}$ for WPP$(\mathcal{R}, \mathcal{X}, \mathcal{S}^t\mathcal{S})$, the numerator of the WPPVE is equal to[5]:

$$|\mathcal{S}\mathcal{X} - \mathcal{S}\mathcal{R}\mathcal{C}|^2 \;=$$

$$\mathit{trace}(\mathcal{X}^t\mathcal{S}^t\mathcal{S}\mathcal{X} - 2\mathcal{X}^t\mathcal{S}^t\mathcal{S}\mathcal{R}\mathcal{C} + \mathcal{C}^t\mathcal{R}^t\mathcal{S}^t\mathcal{S}\mathcal{R}\mathcal{C}) \quad (24a)$$

and the denominator term:

$$|\mathcal{S}\mathcal{X}|^2 \;=\; \mathit{trace}(\mathcal{X}^t\mathcal{S}^t\mathcal{S}\mathcal{X}) \qquad (24b)$$

Notice that the WPPVE depends only on the spectral correlation, $\mathcal{S}^t\mathcal{S}$, of the calibration light spectra. We remind the reader that under MI conditions, $\mathcal{S}^t\mathcal{S} = 0.5\mathcal{I}$. Clearly, substituting the identity for the spectral correlation in (24a) would simplify the equations. However, rather than substituting $\mathcal{I}$ we substitute $\mathcal{P}^t\mathcal{I}\mathcal{P} = \mathcal{P}^t\mathcal{P} = \mathcal{P}$ instead. The reader is reminded that $\mathcal{P}$ is the projector orthogonal to white (see proof in section 3). We are able to make this substitution only because our correction transform $\mathcal{C}$ is white-point preserving. That is, the component of any spectrum in the white direction is perfectly corrected and so we need only consider that part which is orthogonal to white, and so possibly imperfectly corrected, in determining our error measure.

$$|\mathcal{S}\mathcal{X} - \mathcal{S}\mathcal{R}\mathcal{C}|^2 \;=$$

$$0.5\,\mathit{trace}(\mathcal{X}^t\mathcal{P}\mathcal{X} - 2\mathcal{X}^t\mathcal{P}\mathcal{R}\mathcal{C} + \mathcal{C}^t\mathcal{R}^t\mathcal{P}\mathcal{R}\mathcal{C}) \;\equiv\; E \quad (25a)$$

Substituting $\mathcal{S}^t\mathcal{S} = 0.5\mathcal{I}$ in the denominator:

$$|\mathcal{S}\mathcal{X}|^2 \;=\; 0.5\,\mathit{trace}(\mathcal{X}^t\mathcal{X}) \equiv A \qquad (25b)$$

Let us now calculate the WPPVE under MIP conditions. Remember that the MIP spectral correlation $\mathcal{S}^t\mathcal{S} = (1/12)\mathcal{I} + (1/4)\underline{U}\underline{U}^t$. Again, rather than substituting the spectral correlation directly we instead substitute the color signal orthogonal to white:

$$\mathcal{P}[(1/12)\mathcal{I} + (1/4)\underline{U}\underline{U}^t]\mathcal{P} = \mathcal{P}[(1/12)\mathcal{I}]\mathcal{P} = (1/12)\mathcal{P} \quad (26)$$

Substituting (26) in (24a)

$$|\mathcal{S}\mathcal{X} - \mathcal{S}\mathcal{R}\mathcal{C}|^2 \;=$$

$$(1/12)\,\mathit{trace}(\mathcal{X}^t\mathcal{P}\mathcal{X} - 2\mathcal{X}^t\mathcal{P}\mathcal{R}\mathcal{C} + \mathcal{C}^t\mathcal{R}^t\mathcal{P}\mathcal{R}\mathcal{C}) \;=\; (1/6)E \quad (27a)$$

Substituting $(1/12)\mathcal{I} + (1/4)\underline{U}\underline{U}^t = \mathcal{S}^t\mathcal{S}$ in the denominator of 24:

$$|\mathcal{S}\mathcal{X}|^2 \;=\; \mathit{trace}(\mathcal{X}^t[(1/12)\mathcal{I} + (1/4)\underline{U}\underline{U}^t]\mathcal{X}) \quad (27b)$$

Because $\mathcal{X}^t\underline{U} = \underline{1}$ (the observer curves are balanced for white), $\mathit{trace}(\mathcal{X}^t\underline{U}\underline{U}^t\mathcal{X}) = 1 + 1 + 1$. It follows that (27b) can be rewritten as:

$$|\mathcal{S}\mathcal{X}|^2 \;=\; (1/12)A + (3/4) \qquad (27c)$$

We can now relate the Vora Error calculated under MI and MIP conditions. WPPVEMI equals:

$$\text{WPPVEMI} = \frac{E}{A} \quad (28)$$

where E and A are scalar quantities defined in (25a) and (25b). The WPPVEMIP equals:

$$\text{WPPVEMIP} = \frac{(1/6)E}{(A/6)+(3/4)} \quad (29)$$

In (28) and (29), the scalar A depends only on the XYZ sensitivities. For the CIE 1931 2 degree standard observer curves $A = 0.1271875$. Substituting in (29):

$$\frac{\text{WPPVEMI}}{\text{WPPVEMIP}} = \frac{\frac{E}{0.1271875}}{\frac{(1/6)E}{(0.127185/6)+(3/4)}} = 36.38084 \quad (30)$$

The Vora Error calculated under MI conditions when white is preserved is always more than 36 times as large as when MIP conditions are used. That is, assuming that spectra with negative power occur with equal likelihood as all positive spectra, when in fact they can never occur, leads to an error measure which is much larger and so more pessimistic than it ought to be.

5. Conclusion

Much research in color correction is polarized: either one assumes that nothing is known about the colors one will see or everything is assumed to be known. We believe that neither viewpoint is justified: something is always known both about the spectra we will see—they are always all positive—and about the colors we must correct with low error—it is imperative that white looks right. The assumptions of positivity and white-preservation are natural relaxations of the respective positions of total ignorance and total prescience and their adoption has led to improved correction[3, 8]. However, because these ideas have previously been studied in isolation we studied them together in this paper.

We proved a surprising and counterintuitive result: there is no benefit from preserving white and enforcing positivity since the step of preserving white implicitly enforces positivity. We believe the proof of this result is quite important: not only does it help us understand color correction per se but it also helps to explain color correction results reported in the literature. Previous work have shown that the constraints of positivity and preserving white lead to very similar correction performance[8] and now we know that this is as it should be. Moreover, our theoretical argument is used to develop a modified Vora Value measure of filter goodness. The Vora Value measure is a single figure of merit which is commonly used to assess the color correctability of a color device. We show that when white is preserved, this measure is conservative. It always, by a large constant margin, underestimates color correctability. A new more accurate measure is proposed.

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