# Gamut Mapping Based on the Fundamental Components of Reflective Image Specifications

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## Abstract

Gamut mapping transformations are used to convert an image pixel by pixel so that its gamut fits within the gamut of the desired output device. These transformations can be defined independently of the source of illumination by specifying colours in reflectance coordinates, the most economical of which are those based on linear combinations of basis functions. Of course, basis functions differ from one output device to another, making it necessary to map reflectances defined in terms of one basis into another basis. Projective transformations are the most natural way of doing so but are not satisfactory. This paper develops the formal properties of reflective gamut mappings, defines a mapping algorithm based on the fundamental component of the reflectance, and shows that this method consistently conserves colour sensation better than simple projective mappings.

## Introduction

Gamut mapping transformations are used to convert an image pixel by pixel so that its gamut fits within the gamut of the desired output device. These transformations can be defined independently of illumination by specifying colours in reflectance coordinates,[1] the most economical of which are those based on linear combinations of basis functions.[2,3] Usually the basis functions are chosen to minimize the mean square errors between the spectrum of the original reflectances and their representations. Because they are developed empirically they vary from output device to output device, and they depend on the way that the measured colours are sampled. Thus, it is necessary to map reflectances defined in terms of one basis into ones defined in terms of another basis.

Because least squares minimization is linear, linear projective transformations are the natural choice for gamut mapping between reflectance spaces. In this procedure, reflectances are projected orthogonally onto the new basis. Such projective transformations minimize the spectral error but not necessarily the colour difference between original and projected colour. In fact, they can produce unexpectedly large colour differences.

It has long been known that the colour appearances of two reflectances are the same if their fundamental components are the same.[4] This concept can be used to define a new mapping algorithm. Instead of minimizing spectral errors, a mapping that preserves the fundamental component of the reflectance is developed. The results of our experiments show that this method consistently conserves colour sensation better than simple projective mappings.

## Fundamental and Metameric Black Components

For a given illuminant, each surface reflectance can be divided into two components. One is the fundamental component that constitutes our visual sensation, and another is the metameric black component that is invisible to the normal observers.[1] Two reflectances are metameric whenever they have identical fundamental component. Mathematically, each surface reflectance $s$ can be expressed as:

$$s = f_s + b_s$$

where $f_s$ is the fundamental component, $b_s$ is the metameric black of s, and wavelength dependence is left implicit. The fundamental component can be obtained by using a projection operator, $P_f$, which is defined as $A\,(A'A)^{-1}A'$, where $A$ is any matrices of colour mixture functions for the given illuminant.[5] For $M$ be a $N$ by $N$ diagonal matrix that represents the spectral power distribution of the illuminant, and $S$ be a $3$ by $N$ matrix represents the colour matching functions, $A$ is defined as the transpose of the matrix $SM$. The metameric black component can be obtained by using the projection operator $P_b = I - P_f$. Since the matrix $A$ depends on the illuminant, the fundamental component and the metameric black of the surface reflectance are different for different illuminants.

If the reflectances are represented by $N$ sampled points, each reflectance is represented by a point in an $N$ dimensional space. This $N$ dimensional reflectance space is the direct sum of a 3 dimensional subspace spanned by the fundamental components and an $N$-3 dimensional subspace spanned by the metameric black components. The dimension of the fundamental component space depends on the 3 dimensional colour sensation of the human vision system. In the following sections, we refer the former as the fundamental subspace, $F_i$, and the latter as the metameric black subspace, $B_i$, each depending on the illuminant $L_i$.

## Gamut Mapping in Reflectance Spaces

For the colour reproduction under different illuminants, the reflectance spectra of the outputs of colour device are essential for the reproduction process. To reduce the amount of data needed to be processed, the spectra can be represented as a linear combinations of a small number of orthonormal basis vectors. The space spanned by these basis vectors is referred to as a linear reflectance space (LRS) in the following discussion. Several methods[6,7] have been proposed to find the appropriate basis functions. However, the choice of the basis functions is not critical to this study. We used the principal component analysis method to determine the basis functions of the sample reflectance spectra in our experiment.

To map a reflectance $s$, defined in reflectance space $R$, onto the reflectance space $R'$, a simple projective method can be used. The reflectance $s$, which is well-defined in the space $R \cup R'$, is projected onto the basis of $R'$. The resulting projection, $s' = P(s)$ is the reflectance that has the minimum spectral errors according to the metric used to define the basis. Since the measurement of spectral errors are not correlated well with the human colour sensitivity, the mapped reflectance may be objectionably different from the original. To avoid this problem, a mapping which conserves the colour sensation should be used. We next show that a mapping that preserves the fundamental component of the reflectance provides a better result.

## Mapping Based on the Fundamental Components

Let $R$ be the LRS that represents the outputs of a given device. Consider the problem of finding a reflectance $s \in R$ that matches the colour of a reflectance $s'$, defined in a possibly different space $R'$, for the illuminant $L_i$. When $s$ matches with $s'$, their fundamental components are the same under $L_i$. Therefore, the task can be considered as finding a reflectance in $R$ that has the identical fundamental component as $s'$. To determine the fundamental components of the reflectances in $R$, we partition $R$ into two sets, $F_{Ri}$ and $B_{Ri}$. $F_{Ri}$ and $B_{Ri}$ are defined as following:

$$F_{Ri} = \left\{x \mid f_x \neq 0,\, x \in R\right\},$$

$$B_{Ri} = \left\{x \mid f_x \neq 0,\, x \in R\right\},$$

where $f_x \in \mathrm{F}_i$ is the fundamental component of $x$. The set $F_{Ri}$ contains the reflectances (only some of which are physically realizable) that determine the colour sensation and the set $B_{Ri}$ contains the reflectances that are invisible to the normal observer. Note that $F_{Ri}$ is not the fundamental component space intersected with $R$. The partition criterion only ensures that every reflectance belonging to $F_{Ri}$ has a fundamental component and may has a metameric black component in it. $B_{Ri}$ can be obtained by computing the intersection between the reflectance space $R$ and the metameric black subspace $B_i$, and $F_{Ri}$ by computing the difference between $R$ and $B_{Ri}$, that is $F_{Ri} = R - B_{Ri}$. The computation can be carried out by using singular value decomposition (SVD), which allows us to obtain both $F_{Ri}$ and $B_{Ri}$ in a single computation.[8]

For a LRS $R$ that represents a reasonable variety of surface reflectances, $\mathrm{F}_{Ri}$ is a 3 dimensional subspace of $R$. Now let $\{\hat{f}_1, \hat{f}_2, \hat{f}_3\}$ be a basis of $\mathrm{F}_{Ri}$, and let $\hat{f} = f_j + b_j$ for for $j = 1,2,3$, which separates the basis vectors into fundamental and black components. Since $R$ is derived from the outputs of colour device, we expect $R$ can represent the reflectances corresponding to the colours that cover the whole colour space, thus, the span of $\{f_1, f_2, f_3\}$ is the fundamental subspace $F$i for the illuminant $L_i$.

The fundamental component of $s$, $f_s$, can be expressed in terms of $f_i$, that is,

$$f\mathrm{s} = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$$

where the coefficients $\alpha_j$ are the same as the projections of $s$ onto the basis $\{\hat{f}_1, \hat{f}_2, \hat{f}_3\}$. Any element $s' \in \mathrm{F}_{Ri}$ that has the same coefficients $\alpha_i$ with respect to the vectors $\{\hat{f}_1, \hat{f}_2, \hat{f}_3\}$ has the same fundamental component as $s$. That is, $s'$, and $s$ match in colour under illuminant $L_i$. In fact any reflectance $\tilde{s} \in R$ matches $s$ in colour if it can be expressed as:

$$\tilde{s} = a_1 \hat{f}_1 + a_2 \hat{f}_2 + a_3 \hat{f}_3 + b_{\tilde{s}},$$

where $b_{\tilde{s}} \in B_{Ri}$. Among these reflectances, it is possible that the spectral distribution of $\tilde{s}$ may be very different from that of $s$. A large spectral difference is undesirable because the larger the difference in the spectral distribution, the more likely it is that the two reflectances will look different under other illuminants. The reflectance in $R$ that has the same fundamental component with smallest amount of difference in the spectral distribution is a good mapped value for $s$.

To find such reflectance, the residual reflectance, $\Delta s = s - s$, is first computed. Then a spectral distribution similar to is added to $s'$. In order to maintain the fundamental component, the residual reflectance must be chosen from $B_{Ri}$. This can be done by orthogonal projection of $\Delta s$ onto the subspace $B_{Ri}$. The result is a metameric black component in $R$ that has least squared spectral error with $\Delta s$.

## Fundamental Component Mapping under Several Illuminants

Now let us consider the two illuminants case first, which can be easily extended to several illuminants. Let $r \in R$ be one of the reflectances that match the colour of $s$ under the illuminants $L_1$ and $L_2$. Then $r$ can be expressed as:

$$r = f'_{r1} + b_{rl} \qquad \text{for illuminant } L_1$$

and

$$r = f'_{r2} + b_{r2} \qquad \text{for illuminant } L_1$$

where $b_{ri}$ in $B_{Ri}$, and $f'_{ri}$ in $F_{Ri}$ which is same as the fundamental component of $s$ under the illuminant $L_i$. By combining the above two equations, we have

$$\begin{aligned} f'_{r2} - f'_{r1} &= b_{rl} - b_{r2} \\ \Delta f'_{r21} &= P_{b1} \bullet r - P_{b2} \bullet r \\ &= \left(P_{b1} - P_{b2}\right) \end{aligned} \tag{1}$$

where $P_{b1}$, $P_{b2}$ is the linear projection operator that maps $r$ to $b_{r1}$ and $b_{r2}$, respectively. Since the values of $f_{ri}$ and $P_{bi}$ are known once the LRS $R$ and the illuminants are defined, the reflectance $r$ can be solved using (1). Like the single illuminant case, a metameric black component for both illuminants, i.e. $b_{r12} \in (B_{r1} \cap B_{r2})$, can be added to $r$ to reduce the spectral error. It is possible that the linear equation may have no solution for $r$. In such a case SVD can be used to find the least-squares best approximation of $s$.

The above approach can be easily extended to several illuminants. For instance, in the three illuminants case, the left hand side of the Equation (1) can be defined as:

$$\begin{bmatrix} \Delta f'_{r21} \\ \Delta f'_{r32} \end{bmatrix} = \begin{bmatrix} P_{b1} - P_{b2} \\ P_{b2} - P_{b3} \end{bmatrix} \bullet r \, .$$

The reflectance $r$ can be obtained as in the two illuminants case.

## Experimental Results

To test the effectiveness of the fundamental mapping, two LRSs were constructed using principal component analysis. The reflectance samples used to construct the spaces were obtained from 40 real objects, and a set of output colours from a Kodak printer. Two sets of illuminants were used. One set contained the CIE Standard Illuminant A, F7 fluorescent light, and a high pressure sodium light. Another contained the CIE Standard Illuminant D50, D55, and D65. The reflectances of the 40 objects and the 24 samples of the Macbeth Colorchecker were mapped to the two LRSs using the fundamental mapping as well as the simple projection transformation. Average CIELAB colour differences over the set of test reflectances were computed to evaluate the mappings.

The principal angles between the fundamental component subspaces within each set of illuminants have also been computed.[9] They indicate the similarity between two subspaces, providing a qualitative measure of the amount of difference among the light sources of each illuminant set. They are shown in Table (1). As expected, the fundamental component subspaces are much more similar for the second set of illuminants the first set.

**Table 1. Cosine of the Principle Angles Between the Fundamental Component Subspaces of the Pairs of Illuminants within a Set. When the Cosine of Principle Angle Equals to One, it Means Two Vectors in the Two Subspaces Coincides to Each Other.**

| | A vs. F7 | A vs. H.P. Sodium | F7 vs. H.P. Sodium |
|---|---|---|---|
| 1st min. angle | 0.9663 | 0.8945 | 0.8832 |
| 2nd min. angle | 0.9039 | 0.8643 | 0.7195 |
| 3rd min. angle | 0.7818 | 0.5849 | 0.4765 |

| | D50 vs. D55 | D50 vs. D65 | D55 vs. D65 |
|---|---|---|---|
| 1st min. angle | 1.0000 | 1.0000 | 1.0000 |
| 2nd min. angle | 0.9998 | 0.9991 | 0.9997 |
| 3rd min. angle | 0.9994 | 0.9961 | 0.9986 |

Tables 2 and 3 show the average colour differences produced for projection and fundamental component mappings for the two LRSs. As shown in the tables, the fundamental component mapping provides much better results than the simple projection transformation. In most cases, the average colour differences for the fundamental component mapping are 10 times smaller than those for the projective mapping. For individual samples, the projective transformation sometimes maps to reflectances objectionably different from the original, but the fundamental mapping never does.

The fundamental mapping does well for the following reasons. When the viewing illuminants are similar, the fundamental mapping performs effectively due to the overlay of large portion of the fundamental component subspaces for the illuminants (see Table 2b & 3b). Even when the source of illuminations are different, the mapping is still able to find a suitable reflectance that matches the original colour because the fundamental component of the reflectance is similar to the original one for the given illuminants. (see Table 2a & 3a).

**Table 2. The Average CIELab Colour Differences for the Fundamental Mapping (F. M.) and Directional Projection (Proj.) Under Two Sets of Illuminants. The Linear Reflectance Space is Constructed by Using the Reflectances from 40 Real Objects.**

(a)

| Reflectance Samples | CIE A | | F7 | | HP Sodium | |
|---|---|---|---|---|---|---|
| | F. M. | Proj. | F.M. | Proj. | F.M. | Proj. |
| Real Object | 0.29 | 0.35 | 0.01 | 0.43 | 0.02 | 0.49 |
| Macbeth | 0.07 | 0.70 | 0.05 | 1.14 | 0.05 | 1.08 |

(b)

| Reflectance Samples | CIE D50 | | CIE D55 | | CIE D65 | |
|---|---|---|---|---|---|---|
| | F. M. | Proj. | F.M. | Proj. | F.M. | Proj. |
| Real Object | 0.00 | 0.36 | 0.00 | 0.36 | 0.00 | 0.35 |
| Macbeth | 0.00 | 0.81 | 0.00 | 0.81 | 0.00 | 0.81 |

**Table 3. The Average CIELab Colour Differences for the Fundamental Mapping (F.M.) and Directed Projection (Proj.) under Two Set of Illuminants. The Linear Eeflectance Space is Constructed by using the Reflectances of the Evenly Sampled Printer Output Colours**

(a)

| **Reflectance Samples** | **CIE A** | | **F7** | | **HP Sodium** | |
|---|---|---|---|---|---|---|
| | F. M. | Proj. | F.M. | Proj. | F.M. | Proj. |
| **Real Object** | 0.13 | 2.49 | 0.15 | 2.07 | 0.04 | 2.40 |
| **Macbeth** | 0.19 | 1.25 | 0.25 | 1.37 | 0.04 | 2.01 |

(b)

| **Reflectance Samples** | **CIE D50** | | **CIE D55** | | **CIE D65** | |
|---|---|---|---|---|---|---|
| | F. M. | Proj. | F.M. | Proj. | F.M. | Proj. |
| **Real Object** | 0.00 | 2.66 | 0.00 | 2.63 | 0.00 | 2.58 |
| **Macbeth** | 0.00 | 0.94 | 0.00 | 0.90 | 0.00 | 0.84 |

## Conclusions

A gamut mapping based on the fundamental component of the reflectance has been developed. Unlike the projective transformations which minimize spectral errors, this method preserves the fundamental component of the reflectances. As shown in our experimental results, it consistently provides better results than simple projective mappings. In this study, we only considered the mapping between reflectance spaces. For a practical gamut mapping algorithm, the issue of how to handle the out-of-gamut reflectances have to be addressed. We believe that this study will provide a valuable information for the future development of gamut mapping for image specified in reflectance domain.

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