A Minimax Method for Sequential Linear Interpolation of Nonlinear Color Transformations

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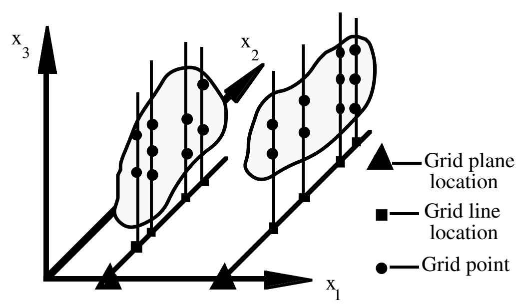
Abstract

We propose a minimax technique to extract the optimum grid structure that will minimize the error in the interpolation of multidimensional functions using sequential linear interpolation (SLI). The error criterion we use is the maximum absolute error. We apply this method to the problem of color printer characterization.

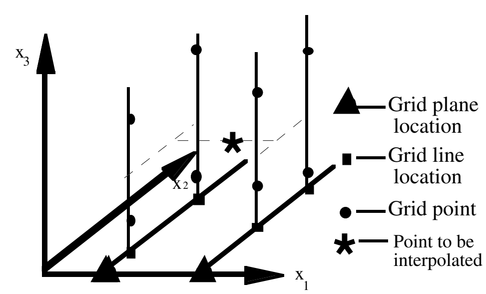
Introduction

Many important problems in image and signal analysis require accurate interpolation of nonlinear multidimensional functions. Recently, an efficient multidimensional function interpolation technique called SLI[1] was proposed. The SLI method uses a structured lookup table with nonuniformly spaced grid points. The SLI grid structure for the interpolation of a 3-D scalar-valued function $f(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3)$ is given in Figure 1a. First, the $x_2 - x_3$ grid planes are placed nonuniformly perpendicular to the x_1 axis. Then, grid lines are placed nonuniformly in the x_2 direction on each grid plane. Lastly, the grid points are placed nonuniformly on each grid line. The grid planes, the grid lines, and the grid points are more densely spaced in regions where the function is more nonlinear. Each function point is trilinearly interpolated from 8 neighboring grid points, 4 from each grid plane on either side of the point to be interpolated. Three nested independent linear interpolations are carried out to find each function point. The interpolation scheme is depicted in Figure 1b.

The SLI structure has two significant advantages. First, it minimizes the need to search for the grid points to be used for interpolation. Second, it enables the development of optimal design procedures. An asymptotic theory which for a fixed number of grid points finds the grid structure that minimizes the mean-squared interpolation error was developed.[1] A similar structure and asymptotic theory was earlier developed for vector quantization.[2,3] In some applications, however, the maximum absolute error is of more concern than is the mean-squared error (MSE). Therefore, we took a minimax approach[4] to the design of SLI structures for surface interpolation. We proposed an iterative minimax grid point allocation method and a method that makes use of asymptotic theory for 2-D function interpolation. The method using the asymptotic design theory gives closed form expressions for the number of grid lines, the optimal grid line density, and optimal grid point density for each grid line in terms of the second order partial derivatives of the 2-D function subject to bounds on the third order derivatives of the function. For functions like color printer transfer functions that exceed these bounds, we have found that the iterative grid structure performs better than the one designed via the asymptotic design theory. In this paper, we generalize the 2-D iterative minimax grid point allocation method to 3-D. We apply this method to the problem of color printer characterization.



(a)



(b)

Figure 1. 3-D SLI grid structure (a) and the 3-D interpolation scheme (b)

Iterative Minimax Grid Point Allocation

Our iterative minimax grid point allocation method is motivated by Kurozomi and Davis's method[5] for finding the best minimax polygonal approximation to a curve that minimizes the maximum distance between the given curve and the approximation, and by Imai and Iri's method[6] for finding a piecewise linear approximation within a given maximum error tolerance to a curve. We first describe the application of this method to 1-D function interpolation and then generalize it to 2-D and to 3-D.

1-D Function Interpolation

In the linear interpolation of the function $f(x)$, $x \in [A, B]$, between N grid points $A = x_1, \ldots x_N = B$, we wish to choose the location of these grid points to minimize the maximum absolute interpolation error. Given any fixed value E for the maximum absolute error, we can find a point distribution that achieves that maximum error by the following recursive procedure for $i = 2,3,\ldots, N$. Assuming that $x_1, \ldots x_i$ have been determined, we move x_{i+1} to the right of x_i until the maximum error in the interval $[x_i, x_{i+1}]$ just equals E. The procedure terminates for some $i = M$, when we pass the right endpoint, *i.e.* $x_M > B$. Setting $x_M = B$ will generally decrease the maximum error in the last interval below the target value E. If $M < N$, we decrease E and repeat the above procedure. If $M > N$, we increase E and repeat it. We iterate until $M = N$.

2-D and 3-D Function Interpolation

Consider the problem of bilinearly interpolating the value of a 2-D function $f(\mathbf{x})$, $\mathbf{x} = (x_1, x_2) \in [A, B] \times [C, D]$, at a point $x_p = (x_{1p}, x_{2p})$ using the 4 neighboring grid points in the SLI grid structure, 2 from each grid line on either side of the point. Since bilinear interpolation is used, it follows from the Cauchy-Schwartz inequality that the absolute interpolation error is bounded above by the sum of the errors for interpolation along x_1 and for interpolation along x_2. Therefore, in our grid design procedure, we bound the target maximum interpolation error E by the sum of the target values for errors E_1 due to interpolation along x_1 and E_2 due to interpolation along x_2. We pick a value in [0,1] for the ratio ρ_1 of E_1 to E. For given values of E_1, ρ_1, and N, we can find the optimal grid structure by the following procedure. First we locate the grid lines along x_1 in a greedy manner as far apart from each other as possible such that the maximum error in each grid line interval just equals $E_1 \equiv \rho_1 E$. Then, we place the grid points onto the grid lines along x_2 using the 1-D procedure described above and the error bound $E_2 \equiv (1-\rho_1)E$. If the number of grid points $M < N$, we decrease E and repeat the above procedure, keeping ρ_1 fixed. If $M > N$, we increase E and repeat it. We iterate until $M = N$. The resulting grid structure is the optimal one for the given number of grid points N and the given ratio ρ_1 of E_1 to E. We repeat this procedure for all possible values of ρ_1; and we find the optimal grid structure that results in minimum maximum interpolation error for the given number of grid points N.

The interpolation of a 3-D function $f(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3) \in [A, B] \times [C, D] \times [F, G]$, is a generalization of the 2-D interpolation in which we bound the target maximum interpolation error E by the sum of the target values for errors E_1 due to interpolation along x_1, E_2 due to interpolation along x_2, and E_3 due to interpolation along x_3. We pick two values in [0,1] for the ratios ρ_1 of E_1 to E and ρ_2 of E_2 to E. We sequentially locate the grid planes, the grid lines onto the grid planes and the grid points onto the grid lines using the target error values $E_1 \equiv \rho_1 E$, $E_2 \equiv \rho_2 E$, and $E_3 \equiv (1 - \rho_1 - \rho_2)E$. For fixed (ρ_1, ρ_2), we obtain the optimal grid structure with N grid points by adjusting E. We iterate over all possible pairs of (ρ_1, ρ_2) until we find the optimal grid structure for the given number of grid points N. The Pidgin ALGOL algorithm given in Figure 2 summarizes the procedure for 3-D function interpolation for a given pair of (ρ_1, ρ_2).

Initialize E
$E_1 = \rho_1 E$
$E_2 = \rho_2 E$
$E_3 = (1 - \rho_1 - \rho_2)E$
$i = 1$, $x_{1i} = A$
while $x_{1i} \le B$
    j = 1, $x_{2ij} = C$
    while $x_{2ij} \le D$
        k = 1, $x_{3ijk} = F$
        while $x_{3ijk} \le G$
            Move $x_{3ij(k+1)}$ to the right of x_{3ijk} until maximum error in
            $\{x_{1i}\} \times \{x_{2ij}\} \times [x_{3ijk}, x_{3ij(k+1)}]$ equals E_3
            $k = k + 1$
        end /*grid points placed onto the jth grid line on the ith grid plane*/
        Move $x_{2i(j+1)}$ to the right of x_{2ij} until maximum error in $\{x_{1i}\} \times [x_{2ij}, x_{2i(j+1)}] \times [C, D] \times [F, G]$ equals E_2
        $j = j + 1$
    end /*grid lines placed onto the ith grid plane*/
    Move $x_{1(i+1)}$ to the right of x_{1i} until maximum error in $[x_{1i}, x_{1(i+1)}$ equals E_1
    $i = i + 1$
end /*grid planes placed*/
if the number of grid points is not N, adjust E and repeat the procedure

Figure 2. Pidgin ALGOL algorithm for the 3-D function interpolation procedure for a given pair of.(ρ_1, ρ_2)

Application to Color Printer Characterization

Modern color management systems require that color printers be characterized in some device independent color space such as CIE L*a*b*. To characterize a printer in the CIE L*a*b* space, we must evaluate the printer transfer function which maps points in the input CMY colorants space to the points in the CIE L*a*b space for every point in the CMY space, *i.e.* every possible colorant combination. Since the complex interaction of the colorants of the printer with the paper substrate makes mathematical modelling of the color printers quite difficult, lookup tables

(LUT's) are used to characterize the color printers. The storage requirement for a LUT including all possible colorant combinations (more than 16 million combinations for a printer that uses 8 bits per colorant) is excessive for most applications. Therefore, only a small selection of the points which form a grid in the input colorants space, are stored in the LUT's; and the function values for the remaining points are interpolated using the table entries.

Many researchers have studied the problems of extracting the optimal grid structure and efficient interpolation. Kasson et al[7], and Hung[8] sampled the printer input (CMY) space uniformly and then used tetrahedral interpolation. Bell and Cowan[9] tessellated the CMY space into tetrahedra and then used tetrahedral interpolation. Kanamori et al[10] and Kotera et al[11] used a uniform grid structure and PRISM interpolation. Bell and Cowan[12] smoothed the data with a tensor product spline before obtaining the SLI grid structure and applying SLI. Chang et al[1] extracted an SLI grid structure using asymptotic design theory and an iterative post-processing technique and then used SLI. In this paper, we use our iterative minimax grid point allocation method to find the optimal SLI grid structure in the CMY domain that will result in the minimum maximum absolute interpolation error and then apply SLI.

Simulation Results

We present simulation results for the interpolation of the printer transfer function obtained from a model[13] for a Xerox color printer. In the high resolution data set generated using the printer model, the printer input CMY space is sampled on a $65 \times 65 \times 65$ uniform grid; and for each grid point, the simulated printer output vector CIE L*a*b* space is calculated. We decompose the Xerox data set into 3 3-D scalar-valued functions of L*, a*, and b* in terms of C, M, and Y.

We present results for the interpolation of 2-D and 3-D functions derived from the Xerox data set. We obtain a 2-D scalar-valued function of L* in terms of M and Y by letting C = 60. In Figure 3, we show, for $N = 100$, the uniform grid structure, Chang et al's minimum mean squared error SLI grid structure,[1] our minimax SLI grid structure, and the resulting interpolation error surfaces. The nonuniform allocation of the grid points in the minimum MSE and the minimax SLI grids improve the error performance noticeably over the uniform grid. The minimax SLI grid structure results in a lower maximum ΔE and a nearly equi-ripple error surface even at the boundaries, whereas the minimum MSE SLI grid structure results in lower RMS ΔE.

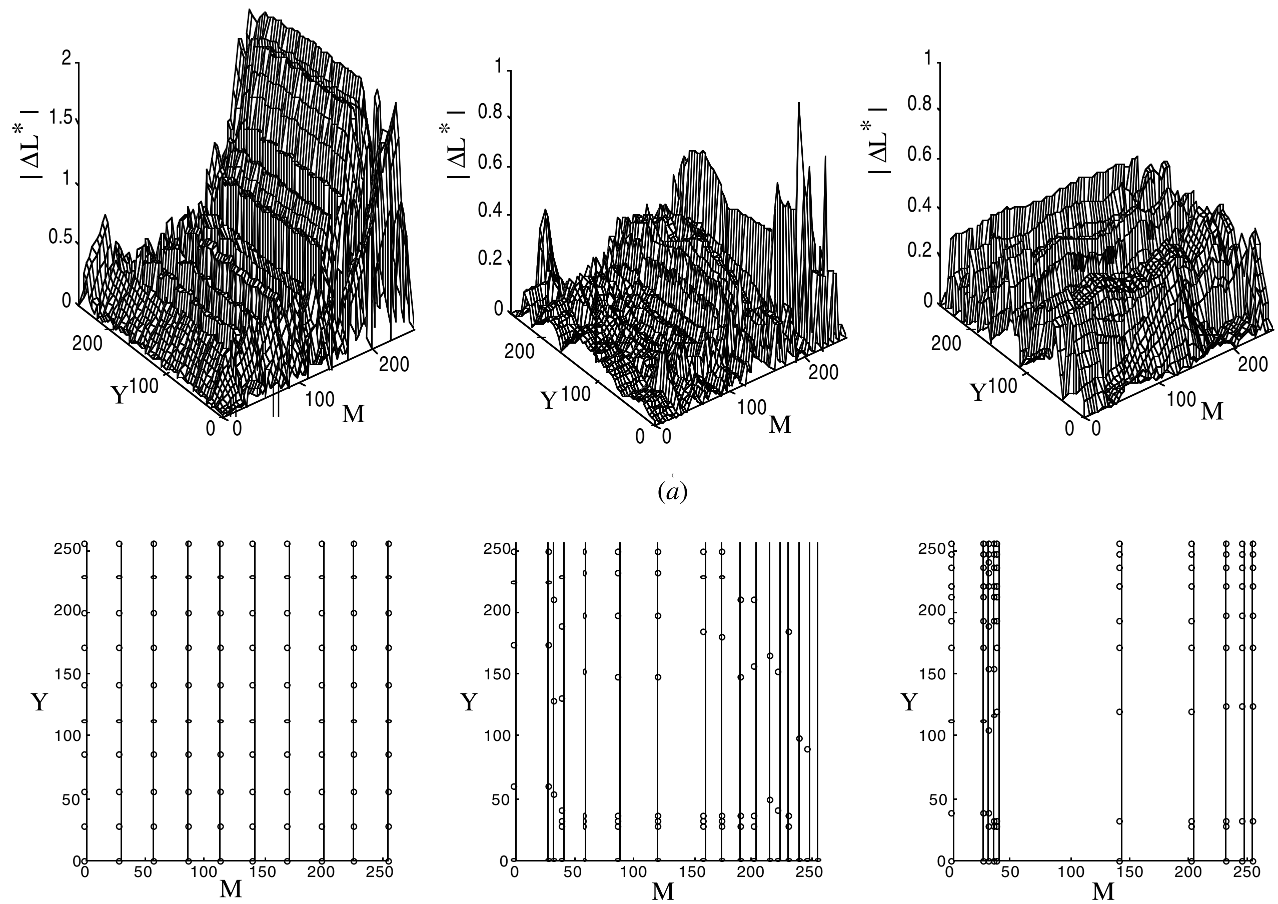
For interpolating the 3-D printer function, we extract 3 separate grid structures for the 3 3-D scalar-valued functions of L*, a*, and b* using the uniform, minimum MSE and iterative minimax grid point allocation methods. We also extract a single grid structure that can be used for the interpolation of all 3 functions by applying our 3-D iterative grid point allocation method jointly to these 3 functions using the maximum ΔE interpolation error as our error criterion where ΔE is given by

$$\Delta E = \sqrt{(\Delta L^*)^2 + (\Delta a^*)^2 + (\Delta b^*)^2}\,. \qquad (1)$$

In Table 1, we compare ΔE error performances for interpolation of the 3-D printer function known at $65 \times 65 \times 65$ points using uniform, minimum MSE SLI, minimax SLI and single grid minimax SLI grid structures with 1000 ($10 \times 10 \times 10$ for uniform), 3375 ($15 \times 15 \times 15$ for uniform) and 15625 ($25 \times 25 \times 25$ for uniform) grid points. Both the minimum MSE and the minimax SLI grid structures perform much better than the uniform ones. The minimax SLI grid structures result in significantly lower maximum ΔE values than the minimum MSE SLI grid structures at the expense of considerably higher relative RMS ΔE values, displaying the tradeoff between maximum error and MSE in the design of grid structures. The single grid minimax SLI grid structures which decrease the storage requirements and the time required to locate the 8 points to be used in the interpolation by a factor of 3 perform almost as well as the minimax grid structures with 3 separate grid structures for L*, a*, and b*.

Table 1. Comparison of grid point allocation methods

No. of Grid Points	Maximum ΔE				RMS ΔE			
	Uniform	Min MSE SLI	Minimax SLI	Minimax SLI (1 grid)	Uniform	Min MSE SLI	Minimax SLI	Minimax SLI (1 grid)
1000	12.0337	5.4120	2.8630	2.8972	2.1912	0.5532	1.0059	1.0258
3375	9.9547	3.1338	2.1188	2.1331	1.8226	0.4570	0.6368	0.6686
15625	7.5808	1.9849	1.0781	1.1541	1.3533	0.1762	0.3626	0.4035



(b)

Figure 3. Comparison of uniform (left), minimum MSE SLI (middle) and minimax SLI (right) error surfaces(a) and grid structures(b). (Note the change in scaling between the error surfaces for the uniform and SLI grid structures.)

Conclusions

In this paper, we developed a minimax method to allocate a fixed number of interpolation grid points to minimize the maximum error in the interpolation of multidimensional functions. We used this technique to extract the SLI grid structure for interpolating a very nonlinear color printer transfer function. Our results indicated that the SLI grid structures perform significantly better than the uniform grid structures. Our simulations also showed that the minimax SLI grid structures result in considerably lower maximum and considerably higher RMS interpolation errors than the minimum MSE SLI grid structures.

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