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On filters making an imaging sensor more colorimetric

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***Abstract*—It is well understood that the color values from a digital camera are functions of the camera's spectral sensitivities, the reflectances of the objects in the scene as well as illumination and any filter that is placed between the object and the sensor. It is vital to select the correct illumination to optimize a color reproduction pipeline. In practice, the choice of the illumination is limited to the spectra of available light sources.**

In this paper, we optimize a camera's colorimetric performance by theoretically mounting a filter to the lens. An ideal spectrum of the filter is obtained using the Luther optimization condition. By using variational calculus we reduce the optimization problem to a system of non linear equations on a Lie group. We solve the system of equations by applying Newton's method on a Lie group with a left invariant Riemannian structure. As expected from the literature, our experiments show quadratic convergence.

A second approach is a redesign of the set-up. This redesign gives us a quadratic optimization problem that is easier to solve. Constraints to this optimization problem gives us control on the transparency of the filter.

I. INTRODUCTION

A. Colorimetry

Camera sensors are designed to capture color images in three channels to imitate the human eye. For different reasons, the spectral sensitivity for the three channels is not equal to the human spectral sensitivity, even after a linear color transform. There are many ways to correct difference of perception between the sensor and the eye. One way is to use many exposures with different filters [3]. As mentioned in the abstract many measures are taken in a camera pipeline to make the picture look good. The method in this paper is not meant to replace the algorithms in the reproduction pipeline but to make a more colorimetric image for the pipeline to work on.

B. Numerical methods on Lie groups

Applications of Lie groups in solving numerical problems have become increasingly popular in the last decades. This was foreseen by Hans Munthe-Kaas at the end of the last century [19]. Lie Groups have proven to be effective in dealing with symmetries in systems of differential equations. Lie group actions have been used to reduce the dimension of differential systems [16], [11], [13]. A reduction of the dimension of a system of PDEs not only reduces the number of parameters, but will also reduce the numerical errors. In differential geometry Lie groups have proved to be a powerful tool in reducing the complexity of problems in physics and mechanics [12] and to search for geometric invariants [1], [22], [23], [21].

In the present paper, we apply Newton's method on Lie groups to solve a minimization problem on $SL(n,\mathbb{R})$. Similar studies have been done on the special orthogonal group $SO(n)$ and the sphere [4].

Mahony et al. state in [17] that there is no natural choice of a left invariant Riemannian structure on a non-compact Lie group. That might be true in a general setting, such as in [23], [2]. However, when the Lie group is a matrix group, such as $SL(3,\mathbb{R})$, there is a natural choice. We will choose the metric that is equal to the Frobenius Norm at the identity. This norm has minimal condition number and is therefore numerically favorable.

C. Imaging science

There are problems related to photography that are not addressed by making the camera more colorimetric. The light condition will often be unpredictable when capturing an image. Therefore, it is necessary to use a color correction algorithm [10], [5]. This is discussed in [6].

The human retina has three types of cones. These are photo sensitive cells that enable us to sense colors. Simply explained, these three cone types are responsible for sensing light with long, medium and short wavelength respectively. The quantified responses (L, M, S) from these three cone types give points in an LMS color space. The three different cone types have their sensitivity peaks in the long, medium, or short wavelength part of the visible spectrum.

The first quantitative experiments measuring the human eye's response to monochromatic light were done in 1931 [24]. Test persons observed a series of monochromatic light with the wavelength λ ranging from 380nm to 780nm. For each wavelength, the test persons matched the observed light with a mixture of red, green and blue light, quantified with real numbers R, G and B respectively. The triple (R, G, B) is referred to as the tristimulus of the observed light in the so-called CIE 1931 RGB color space. In the so-called CIE 1931 XYZ color space the same tristimulus is written as a linear combination (X, Y, Z) of (R, G, B). The experiment resulted in the so-called CIE-XYZ color matching functions (CIE CMF) $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, and $\bar{z}(\lambda)$.

In general, a light source is not monochromatic but a mixture of photons of all wavelengths. The light source is characterized by its total energy $I(\lambda)$ per unit wavelength. The tristimulus values for a standard observer can be calculated by $X = k\int I(\lambda)\bar{x}(\lambda)\,\mathrm{d}\lambda$, $Y = k\int I(\lambda)\bar{y}(\lambda)\,\mathrm{d}\lambda$, and $Z = k\int I(\lambda)\bar{z}(\lambda)\,\mathrm{d}\lambda$, where k is a constant. We will write the CIE 1931 XYZ sensitivity function as $\mathbf{s}_1(\lambda) = \begin{bmatrix}\bar{x}(\lambda) & \bar{y}(\lambda) & \bar{z}(\lambda)\end{bmatrix}^T$. Let $\mathbf{s}_2(\lambda) = \begin{bmatrix}r(\lambda) & g(\lambda) & b(\lambda)\end{bmatrix}^T$ be the RGB sensitivity function for a camera sensor.

Finlayson et al. [6] had the ingenious idea of designing an optical filter for a digital camera so that the camera mimics the XYZ response. They use the technique of Alternating Least Square, (ALS). The performance of the ALS algorithm is not discussed or measured in [6] for their specific application.

We found that their algorithm converged linearly with low convergence rate. Their method required more than thousand iterations where the Lie group method required less than ten iterations for the same accuracy.

In the section, we give an alternative and simpler way of solving Finlayson's original problem. By using calculus of variations we eliminate the filter from the equations. This gives a deeper insight into the original problem of Finlayson et al. by showing that their problem is equivalent to finding a color transform matrix A so that $\mathbf{s}_1(\lambda)$ and $A\,\mathbf{s}_2(\lambda)$ are "as parallel as possible", which will be more clear in Section II-B.

In Section III, we give a much simpler version of the original problem in [6]. Instead of applying a linear color transform of the camera output, we apply an energy preserving color transform of the XYZ sensitivity functions. Elimination of the filter from the equations reduces the problem to the question of finding the smallest eigenvector of a positive definite symmetric 9×9 matrix. We used the Rayleigh's quotient iteration method which converges quadratically. The method is therefore very fast.

In Section IV we present the experiment results for the original problem and the simplified problem. There are different metrics for colorimetric in literature. In the discussion, we use the Vora-value [26] and the normalized spectral root-mean-square errors (NRMSE). The conclusion from the comparison of the original problem and the simplified problem is that the simplified problem gives a slightly better performance with respect to the Vora-value metric. As expected, the original problem scores better for NRMSE.

In Section V and VI we add constraints to the filter in the simplified problem. We solve the problem by using quadratic programming. In addition we also compare the Vora-value for different constraints, the simplified problem and the original problem.

II. Mimicking XYZ by using an optical filter and a linear color space transform

A. Stating the original problem

In this section we revisit the original idea of Finlayson et al. Their idea was to fit an optical filter with the spectral transparency function $t(\lambda)$ together with a color conversion matrix A so that the color responses from the camera and the human eye are as equal as possible. The value $t(\lambda)$ is the **transparency** of the filter for monochromatic light of wave length λ, i.e. $100 \cdot t(\lambda)$ percent of photons with wave length λ will pass through the filter.

The original idea of Finlayson et al. was to minimize the following functional

$$W_1(A,t) = \int_{\lambda_{\min}}^{\lambda_{\max}} \|\mathbf{s}_1(\lambda) - t(\lambda)A\,\mathbf{s}_2(\lambda)\|^2 \,\mathrm{d}\lambda. \tag{1}$$

A constraint on A is convenient and we will come back to it later.

Our contribution is twofold. **One:** We show that their problem can be reduced to the more general question of finding a "best" color space transform matrix with respect to another criteria that will be shown in Section II-B.

Two: We solve this problem by using Newton's method on Lie groups. There are two available variants of the Newton methods on Lie groups. One method depends on the Lie group structure only [20]. This method converges quadratically for classical problems [20]. The other method uses the left invariant metric and its corresponding Levi Cevita covariant derivative [18].

Experiments show a very fast convergence for both methods. The latter variant of Newton's method is less sensitive to the choice of a first guess. The first variant required other methods such as the deepest decent method to converge in the experiment. The first guess was obtained from the simplified problem which is explained in Section III.

B. Transforming the original problem

Assume that A is 3×3 matrix so that $A\,\mathbf{s}_2(\lambda) \neq \mathbf{0}$ for all $\lambda \in [\lambda_0, \lambda_1]$. For any such fixed A, calculus of variations gives that the minimum of the functional (1) is obtained when

$$t(\lambda) = \frac{\mathbf{s}_2(\lambda)^T A^T \mathbf{s}_1(\lambda)}{\mathbf{s}_2(\lambda)^T A^T A\,\mathbf{s}_2(\lambda)}. \tag{2}$$

We eliminate $t(\lambda)$ in (1) by using the formula in equation (2).

$$W_1(A) = \int_{\lambda_{\min}}^{\lambda_{\max}} \mathbf{s}_1(\lambda)^T (I - P_2(A))\mathbf{s}_1(\lambda)\,\mathrm{d}\lambda \tag{3}$$

where $P_2(A,\lambda) = \frac{A\mathbf{s}_2(\lambda)\mathbf{s}_2(\lambda)^T A^T}{\mathbf{s}_2(\lambda)^T A^T A\mathbf{s}_2(\lambda)}$ is the projection onto $A\mathbf{s}_2(\lambda)$. Minimizing $W_1(A)$ is equivalent to maximizing the functional

$$F(A) = \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{f(A)}{h(A)} \mathrm{d}\lambda \tag{4}$$

where $f(A) = (\mathbf{s}_1^T A\,\mathbf{s}_2)^2$ and $h = \mathbf{s}_2^T A^T A\,\mathbf{s}_2$, as functions of λ. This is equivalent to looking for a matrix A such that the energy of $\mathbf{s}_1(\lambda) \cdot \frac{A\mathbf{s}_2(\lambda)}{|A\mathbf{s}_2(\lambda)|}$ is as large as possible. Notice that any scalar expansion of A will not change $f(A)/h(A)$. We may therefore introduce a constraint on A.

C. The constraint

The most beautiful constraint is to require that A be an element in the special linear group $SL(3,\mathbb{R}) = \{A \in GL(3,\mathbb{R}) \mid \det A = 1\}$. We recall that the elements of $GL(3,\mathbb{R})$ are the invertible real 3×3-matrices.

A 3×3 matrix X is a tangent vector at an element A in $SL(3,\mathbb{R})$ if there exists a curve $\gamma(t)$ in $SL(3,\mathbb{R})$ with $\gamma(0) = A$ and $\gamma'(0) = X$. Let X and Y be two tangent vectors at A in $SL(3,\mathbb{R})$. The standard left invariant metric on $SL(3,\mathbb{R})$ is defined by

$$g(X,Y) = \operatorname{trace}\Big((A^{-1}X)^T A^{-1} Y\Big). \tag{5}$$

Remark 1: The standard metric is a choice among infinitely many left invariant metrics. In fact, $g_S(X,Y) = \operatorname{trace}\Big((A^{-1}X)^T S A^{-1} Y\Big)$ is also a left invariant metric whenever S is a positive definite[1] symmetric 3×3-matrix. The choice $S = I$ has condition number 1.

The Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ is the set of the tangent vectors at the identity matrix I in $SL(3,\mathbb{R})$. $\mathfrak{sl}(3,\mathbb{R}) = \{X \mid \operatorname{trace} X = 0\}$.

[1]A symmetric matrix is **positive definite** if it has only positive eigenvalues.

Given $X \in \mathfrak{sl}(3,\mathbb{R})$, define the left invariant vector field $\tilde{X}$ with value AX at the point A. The elements in the Lie algebra $\mathfrak{sl}(3,\mathbb{R}) = \{X \,|\, \operatorname{trace} X = 0\}$ are usually identified with their corresponding left invariant vector fields.

Let $X_1, \ldots, X_8$ be an orthonormal basis of the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ under the standard left invariant metric on $SL(3,\mathbb{R})$.

$$X_1 = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0&0&0\\0&0&0\\1&0&0 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0&0&0\\0&0&0\\0&1&0 \end{bmatrix}, \qquad (6)$$

$$X_7 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix}, \quad X_8 = \frac{\sqrt{6}}{6}\begin{bmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{bmatrix}.$$

There is an exponential map [25, pp. 89–90] that sends $X \in \mathfrak{sl}(3,\mathbb{R})$ to the matrix $\exp X \in SL(3,\mathbb{R})$ defined by $\exp X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$.

Denote derivation of f in A along the curve $C(t) = A\exp(tX)$ through A as $[\tilde{X}f](A)$. That is

$$[\tilde{X}f](A) = \left.\frac{d}{dt}\right|_{t=0} f(A\exp(tX)).$$

In fact, $\tilde{X}$ is the **left invariant vector field generated by** $X \in \mathfrak{sl}(3,\mathbb{R})$. This notation is used widely in literature.

D. Finding the maximum of $F(A)$

The goal is to solve $[\tilde{X}_j F](A) = 0$ for $i = 1, 2, \ldots, 8$. That is, to find a critical point for the maximization problem.

Taylor's formula on Lie groups [9, page 105] gives

$$f(A\exp(X)) = \sum_{k=0}^{\infty} \frac{1}{k!}[\tilde{X}^k f](A).$$

For small X we have the approximation $f(A\exp(X)) \approx f(A) + [\tilde{X}f](A)$. By setting $f = \tilde{X}_j F$, the linearized problem is therefore $[\tilde{X}_i F](A\exp(X)) \approx [\tilde{X}_i F](A) + [\tilde{X}\tilde{X}_i F](A)$.

E. The derivatives

Derivation of f and h gives $\tilde{X}_i f(A) = 2\mathbf{s}_1^T A X_i \mathbf{s}_2 \mathbf{s}_1^T A \mathbf{s}_2$ and $\tilde{X}_i h(A) = 2\mathbf{s}_2^T (A^T A X_i)\mathbf{s}_2$. The derivative of $G(A) = \frac{f(A)}{h(A)}$ is

$$\tilde{X}_i G = \frac{\tilde{X}_i f(A)}{h(A)} - \frac{f(A)\tilde{X}_i h(A)}{h(A)^2}.$$

We find the second derivatives of f and h.

$$\tilde{X}_i\tilde{X}_j f(A) = 2\mathbf{s}_1^T A X_i X_j \mathbf{s}_2 \mathbf{s}_1^T A \mathbf{s}_2 + 2\mathbf{s}_1^T A X_j \mathbf{s}_2 \mathbf{s}_1^T A X_i \mathbf{s}_2$$
$$\tilde{X}_i\tilde{X}_j h(A) = 2\mathbf{s}_2^T (A^T A X_i + X_i^T A^T A) X_j \mathbf{s}_2.$$

The values of the derivatives in I are

$$\begin{aligned} \tilde{X}_i f(I) &= 2\mathbf{s}_1^T X_i \mathbf{s}_2 \mathbf{s}_1^T \mathbf{s}_2 \\ \tilde{X}_i h(I) &= 2\mathbf{s}_2^T X_i \mathbf{s}_2 \\ \tilde{X}_i\tilde{X}_j f(I) &= 2\mathbf{s}_1^T X_i X_j \mathbf{s}_2 \mathbf{s}_1^T \mathbf{s}_2 + 2\mathbf{s}_1^T X_j \mathbf{s}_2 \mathbf{s}_1^T X_i \mathbf{s}_2 \\ \tilde{X}_i\tilde{X}_j h(I) &= 2\mathbf{s}_2^T (X_i + X_i^T) X_j \mathbf{s}_2. \end{aligned}$$

The second derivative of $G(A) = \frac{f(A)}{h(A)}$ in I is

$$\tilde{X}_i\tilde{X}_j G(I) = \frac{\tilde{X}_i\tilde{X}_j f(I)}{h(I)} - \frac{\tilde{X}_i f(I)\tilde{X}_j h(I)}{h(I)^2}$$
$$- \frac{\tilde{X}_j f(I)\tilde{X}_i h(I)}{h(I)^2} - \frac{f(I)\tilde{X}_i\tilde{X}_j h(I)}{h(I)^2} + 2\frac{f(I)\tilde{X}_i h(I)\tilde{X}_j h(I)}{h(I)^3}$$

F. The covariant method

Let $g(\bullet,\bullet)$ be the standard left invariant metric on $SL(3,\mathbf{R})$ defined in equation (5). The Levi Cevita connection ∇ is then defined by

$$g(\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}) = \tfrac{1}{2}\big(g([\tilde{X},\tilde{Y}],\tilde{Z}) - g([\tilde{X},\tilde{Z}],\tilde{Y}) - g([\tilde{Y},\tilde{Z}],\tilde{X})\big). \qquad (7)$$

The gradient is $\operatorname{grad} F = \sum \tilde{X}_i F X_i$ since the frame is orthogonal. The covariant linearization of the problem is

$$L_{\exp(-X)*}\Big(\operatorname{grad} F(\exp(X))\Big)^2 \approx \operatorname{grad} F(I) + \operatorname{Hess}_F(X),$$

where $\operatorname{Hess}_F(\tilde{X}) = \nabla_{\tilde{X}} \operatorname{grad} F$ is the Hessian tensor of F. We will use the following formula to calculate the Hessian tensor. $\operatorname{Hess}_F(\tilde{X}_i, \tilde{X}_j) = \tilde{X}_i\tilde{X}_j F - g(\nabla_{\tilde{X}_i}\tilde{X}_j, \operatorname{grad} F)$. This concludes this section. The formulas are implemented in MATLAB and the results from the experiments are presented in Section IV. We used algorithm 1 with $tolerance = 10^{-12}$.

Algorithm 1: Covariant method.

Set $A_0 =$ initial guess.
Do
  $F_i(A) = F(AA_i)$
  Solve $\operatorname{grad} F_i(I) + \operatorname{Hess}_{F_i}(X) = \mathbf{0}$
  $A_{i+1} = \exp(X)A_i$
  i:=i+1
until $||A_i - A_{i-1}|| < tolerance$

III. A SIMPLIFIED VERSION OF THE PROBLEM

A. A reformulation of the problem

In the former section we solved the problem where a combination of an optical filter and a linear color transform gave sensitivity functions for the camera as close as possible to the human XYZ-sensitivity functions. In this section we propose a problem, where we search an optical filter for the camera that gives sensitivity functions as close as possible to some linear color transform of the human XYZ-sensitivity functions. In Section IV we will show that the simplified problem gives very good results with respect to the Vora-value.

To be precise, we minimize the functional

$$W_2(A,t) = \int_{\lambda_{\min}}^{\lambda_{\max}} \|A\,\mathbf{s}_1(\lambda) - t(\lambda)\mathbf{s}_2(\lambda)\|^2 \,\mathrm{d}\lambda \qquad (8)$$

under the carefully designed constraint

$$C(A) = \int_{\lambda_{\min}}^{\lambda_{\max}} \|A\mathbf{s}_1(\lambda)\|^2 - \|\mathbf{s}_1(\lambda)\|^2 \,\mathrm{d}\lambda = 0 \qquad (9)$$

on A. This constraint ensures that for each value of λ, $A\mathbf{s}_1(\lambda)$ has length close to $\mathbf{s}_1(\lambda)$.

[2] L_A is left-multiplication with A in $SL(3,\mathbb{R})$ and L_{A*} is the differential of L_A. If $X = \gamma'(0)$ is a vector at a point B in $SL(3,\mathbb{R})$, then $L_{A*}X$ is the vector at $L_A B = AB$ given by $L_{A*}X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Big(A\gamma(t)\Big)$. More about the above can be found in [15, p. 8].

B. Eliminating the filter and solving the problem

Given an A, calculus of variations gives that

$$t(\lambda) = \frac{\mathbf{s}_2(\lambda)^T A\,\mathbf{s}_1(\lambda)}{\mathbf{s}_2(\lambda)^T \mathbf{s}_2(\lambda)} \tag{10}$$

minimizes the functional (8). Elimination of $t(\lambda)$ in (8) by using equation (10) gives

$$W_2(A) = \int_{\lambda_{\min}}^{\lambda_{\max}} \|(I - P_2(\lambda))A\,\mathbf{s}_1(\lambda)\|^2 \,\mathrm{d}\lambda \tag{11}$$

where $P_2(\lambda) = \frac{\mathbf{s}_2(\lambda)\mathbf{s}_2(\lambda)^T}{\mathbf{s}_2(\lambda)^T\mathbf{s}_2(\lambda)}$. For each λ, $P_2(\lambda)$ is the projection onto $\mathbf{s}_2(\lambda)$. The matrix A that minimizes $W_2(A)$ under the constraint $C(A) = 0$ is found by using the Lagrange method. Let $A = [a_{ij}]$ and $E_{ij} = \frac{\partial A}{\partial a_{ij}}$, $i, j = 1, 2, 3$. The derivatives of $W_2(A)$ and $C(A)$ are

$$\frac{\partial W_2(A)}{\partial a_{ij}} = \int_{\lambda_{\min}}^{\lambda_{\max}} 2\,\mathbf{s}_1(\lambda)^T E_{ij}^T (I - P_2(\lambda))A\,\mathbf{s}_1(\lambda)\,\mathrm{d}\lambda \tag{12}$$

and

$$\frac{\partial C(A)}{\partial a_{ij}} = \int_{\lambda_{\min}}^{\lambda_{\max}} 2\,\mathbf{s}_1(\lambda)^T E_{ij}^T A\,\mathbf{s}_1(\lambda)\,\mathrm{d}\lambda. \tag{13}$$

Notice that (12) and (13) are linear in A and that we can write $\nabla W_2(A) = H\mathbf{x}$ and $\nabla C(A) = K\mathbf{x}$ where H and K are positive definite symmetric 9×9 matrices and $\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{33} \end{bmatrix}^T \in \mathbb{R}^9$.

The Lagrangian for this problem is $\mathcal{L}(A, \mu) = W_2(A) - \mu C(A)$. The Lagrange multiplier equations are $H\mathbf{x} = \mu\, K\mathbf{x}$. Notice that this is an eigenvalue problem $K^{-1}H\mathbf{x} = \mu\mathbf{x}$. It can be proved that μ has to be the lowest eigenvalue.

Therefore we can use Rayleigh's quotient iteration method.

IV. Comparison between the original problem and the simplified problem

We run an experiment where we solve the problems in Section II and Section III with spectral sensitivity data extracted from a wide range of cameras. We use sensitivity data from RIT, measured by Gu Jinwei et al. [8], [7]. We have omitted cameras where the brands are under-represented in the data to avoid being unfair to brands based on a few samples.

We calculated the normalized root mean square error (NRMSE) between the reference CIE CMF's and the filter adjusted camera sensitivity functions. See [6] for reference. Table I shows the results of the experiment for 19 of the cameras in the RIT database. The solution from using the Lie group method gave the same values for NRMSE as the method of Finlayson et al.

The following definition of the Vora-value is equivalent to the definition in [26], [5].

$$v = \tfrac{1}{3}\,\mathrm{trace}\left({M_{11}}^{-1} M_{12} {M_{22}}^{-1} M_{21}\right), \tag{14}$$

where $M_{ij} = \int_{\lambda_{\min}}^{\lambda_{\max}} \mathbf{s}_i(\lambda)^t \mathbf{s}_j(\lambda)\,\mathrm{d}\lambda$. Table II shows the Vora-values [26] from the experiments. The results show that the optimal filters for both the original problem and the simplified problem increase the Vora-values significantly for all cameras in the test data. However the Vora-values do not differ significantly between the two problems. For ten of the cameras, the simplified method gives a slightly higher Vora-value than Finlayson et al.'s problem. Though, Finalyson et al.'s method is slightly better for nine of the cameras.

Another important fact is the significant difference between the transparency functions for the best filters in both our method and Finlayson et al.'s method. We can see this clearly by comparing Figure 1(a) with Figure 3(a). The best filter for the Nikon D50 is much more transparent than the best filter for Canon 50D. Partly for that reason, Finlayson et al. can not conclude with the following from their data:

> "Taken as a group, Canon cameras can be filter-corrected to become colorimetric more readily than Nikon cameras. The corrected Canon cameras have Vora-Values of 0.972 at least, with an average value as high as 0.987 for the whole subset compared to 0.944 of Nikon subgroup [6]."

Moreover, our results in Section VI show that the Nikon cameras can have significantly higher Vora-numbers than claimed by Finlayson et al.

V. Controlling the transparency of the filter

We control the filter transparency $t(\lambda)$ with two conditions. The first and obvious condition is that

$$0 \leq t(\lambda) \leq 1. \tag{15}$$

A second condition is some weighted integral constraint

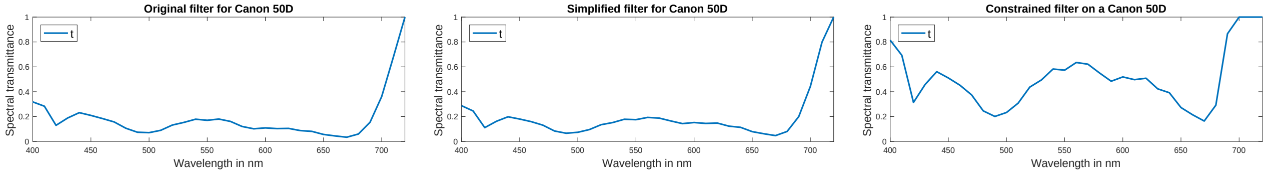
$$\int_{\lambda_{\min}}^{\lambda_{\max}} w(\lambda)(t(\lambda) - t_0)\,\mathrm{d}\lambda = 0, \tag{16}$$

where $0 < t_0 < 1$. In this section we will require the filter to be faithful to the perceived relative luminance Y in CIE D65 daylight illuminant I_{D65} [14]. That is, $w(\lambda) = I_{D65}(\lambda)Y(\lambda)$. These constraints can be used with both the quartic functional $W_1(A, t)$ in (1) or the quadratic functional $W_2(A, t)$ in (8). These two give almost the same Vora-value for their minimal solutions as have been shown in Section IV.

The constrained problem formed by problem (1) and the constraints (15) and (16) are hard to solve. The solution of this problem is left as a hard exercise. On the other hand, the problem formed by the functional (8) and the constraints (15) and (16) is a quadratic programming problem when discretized.

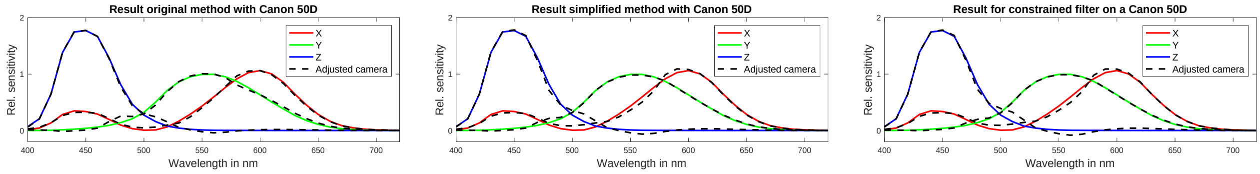
VI. Results for constrained filters

We ran the experiment where we solved the constrained problem in Section V with the values $t_0 = 0.1, 0.2, \ldots, 0.8, 0.9, 0.95, 0.98, 0.99$. The results for $t_0 = 0.5$ are shown in the figures 1(c), 2(c), 3(c), and 4(c). The Vora-values are displayed in Table II. The NRMSE are shown in Table I. Figure 5 shows the Vora-values as functions of t_0 for the Nikon and Canon cameras.



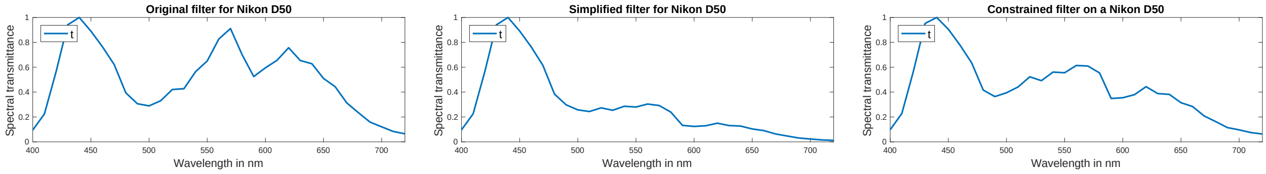
(a) Filter solved for the original problem. (b) Filter solved for the simplified problem. (c) Filter solved for the constrained problem.

Fig. 1. Spectral transparency functions for filters adapted for a Canon 50D.



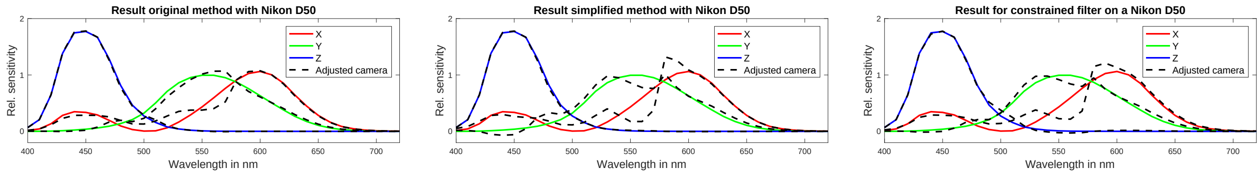
(a) Comparison for the original problem. (b) Comparison for the simplified problem. (c) Comparison for the constrained problem.

Fig. 2. The figures show the transformed sensitivity functions for a Canon 50D compared to the CIE CMFs.



(a) Filter solved for the original problem. (b) Filter solved for the simplified problem. (c) Filter solved for the constrained problem.

Fig. 3. Spectral transparency functions for filters adapted for a Nikon D50.



(a) Comparison for the original problem. (b) Comparison for the simplified problem. (c) Comparison for the constrained problem.

Fig. 4. The figures show the transformed sensitivity functions for a Nikon D50 compared to the CIE CMFs.

Camera	None.	Lie group	Simpl.	Constr.
Canon $1D_{MarkIII}$	0.309	0.056	0.075	0.075
Canon 20D	0.332	0.069	0.090	0.124
Canon 300D	0.324	0.091	0.132	0.102
Canon 40D	0.298	0.062	0.079	0.093
Canon 500D	0.249	0.057	0.072	0.082
Canon 50D	0.250	0.052	0.067	0.074
Canon $5D_{MarkII}$	0.285	0.047	0.059	0.064
Canon 600D	0.261	0.058	0.073	0.083
Canon 60D	0.256	0.062	0.079	0.090
Nikon D200	0.304	0.107	0.194	0.156
Nikon D3	0.298	0.106	0.183	0.152
Nikon D300s	0.324	0.110	0.208	0.152
Nikon D3X	0.302	0.112	0.190	0.156
Nikon D40	0.305	0.104	0.173	0.147
Nikon D50	0.318	0.109	0.180	0.155
Nikon D5100	0.329	0.111	0.193	0.209
Nikon D700	0.292	0.106	0.185	0.153
Nikon D80	0.374	0.104	0.196	0.159
Nikon D90	0.317	0.113	0.208	0.156

TABLE I
THE NRMSE BETWEEN THE CORRECTED CAMERA SENSITIVITIES AND THE CIE CMF'S.

Camera	None.	Lie group	Simpl.	Constr.
Canon $1D_{MarkIII}$	0.9294	0.9861	0.9894	0.9890
Canon 20D	0.9218	0.9814	0.9842	0.9784
Canon 300D	0.9272	0.9720	0.9719	0.9802
Canon 40D	0.9321	0.9860	0.9891	0.9870
Canon 500D	0.9481	0.9911	0.9922	0.9909
Canon 50D	0.9496	0.9921	0.9929	0.9920
Canon $5D_{MarkII}$	0.9342	0.9926	0.9941	0.9936
Canon 600D	0.9491	0.9907	0.9924	0.9910
Canon 60D	0.9496	0.9895	0.9913	0.9898
Nikon D200	0.9267	0.9441	0.9376	0.9578
Nikon D3	0.9313	0.9462	0.9439	0.9607
Nikon D300s	0.9200	0.9442	0.9233	0.9567
Nikon D3X	0.9270	0.9416	0.9388	0.9571
Nikon D40	0.9237	0.9474	0.9513	0.9627
Nikon D50	0.9176	0.9424	0.9487	0.9596
Nikon D5100	0.9185	0.9443	0.9338	0.9417
Nikon D700	0.9333	0.9458	0.9424	0.9600
Nikon D80	0.8881	0.9454	0.9382	0.9569
Nikon D90	0.9221	0.9413	0.9259	0.9563

TABLE II
THE VORA-VALUES FOR THE CAMERA SENSITIVITIES WITH RESPECT TO THE CIE CMF'S.

VII. Concluding remarks

A. Performance

The simplified method used 2.3 ms in average for the cameras in the experiments. The constrained filter method used 3.8 ms in average to run. The Lie group method for solving the original problem used 34 ms in average. Our implementation of the ALS method used by Finlayson et al. [6] used 225 ms in average. The Lie group method and the ALS method gave the same solution. All calculations were done in MATLAB on a 2.2 GHz Intel Core i7 CPU.

B. Suggestion for future work

The original idea of Finlayson et al. is a really good one and it deserves to be studied further. The results in this paper are just touching the surface of the these type of problems. Maximizing the Vora-value could be an interesting task for future research.

C. Practical applications

It is impractical to make some of the filters that are solutions to the problems. The state of the art in filter technology gives constraints on what is possible in filter manufacturing. A future work could be designing real world filters with the suggested properties.

D. Disclaimer

The results in this paper consider two top brands of digital cameras. Although, one of the brands scores better in these calculations, we will emphasize that these calculations are theoretical. As long as these filters do not exist in real life, the results in this paper should not be used as an argument for choosing one camera over the other.

References

[1] Agaoka, Y.: On the curvature of Riemannian submanifolds of codimension 2. Hokkaido Mathematical Journal **14**, 107–135 (1985)

[2] Agaoka, Y., Hashinaga, T.: On local isometric embeddings of three-dimensional Lie groups. Geometriae Dedicata (2019). DOI 10.1007/s10711-019-00472-y

[3] Farrell, J., Wandell, B.: (U.S. Patent No. 5479524, Washington, DC: U.S. Patent and Trademark Office, 1995.)

[4] Ferreira, R., Xavier, J., P., C.J., Barroso, V.: Newton method for Riemannian centroid computation in naturally reductive homogeneous spaces. In: 2006 IEEE International Conference on Acoustics Speech and Signal Processing Proceedings, vol. 3, pp. III–III (2006). DOI 10.1109/ICASSP.2006.1660751

[5] Finlayson, G.D., Mackiewicz, M., Hurlbert, A.: Color correction using root-polynomial regression. IEEE Transactions on Image Processing **24**(5), 1460–1470 (2015). DOI 10.1109/TIP.2015.2405336

[6] Finlayson, G.D., Zhu, Y., Gong, H.: Using a simple colour pre-filter to make cameras more colorimetric. In: Proceedings of the 26th Color and Imaging Conference, pp. 182–186 (2018)

[7] Gu, J.: What is the space of spectral sensitivity functions for digital color cameras. http://www.gujinwei.org/research/camspec/ (2013)

[8] Gu, J., Jiang, J., Susstrunk, S., Liu, D.: What is the space of spectral sensitivity functions for digital color cameras? In: Proceedings of the 2013 IEEE Workshop on Applications of Computer Vision (WACV), WACV '13, pp. 168–179. IEEE Computer Society, Washington, DC, USA (2013)

[9] Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Pure and Applied Mathematics. Academic Press (1978)

[10] Hong, G., Luo, M.R., Rhodes, P.A.: A study of digital camera colorimetric characterization based on polynomial modeling. Color Research & Application **26**(1), 76–84 (2001)

[11] Hsiang, W.Y., Straume, E.: Global geometry of 3-body motions with vanishing angular momentum I. Chinese Annals of Mathematics, Series B **29**(1), 1–54 (2008). DOI 10.1007/s11401-007-0153-8

[12] Ibragimov, N.K.: Group analysis of ordinary differential equations and the invariance principle in mathematical physics (on the occasion of the 150th anniversary of the birth of Sophus Lie). Uspekhi Mat. Nauk **47**(4(286)), 83–144, 235 (1992). DOI 10.1070/RM1992v047n04ABEH000916

[13] Iserles, A., Munthe-Kaas, H.Z., Nørsett, S.P., Zanna, A.: Lie-group methods. Acta Numerica **9**, 215—-365 (2000). DOI 10.1017/S0962492900002154

[14] Judd, D.B., MacAdam, D.L., Wyszecki, G., Budde, H.W., Condit, H.R., Henderson, S.T., Simonds, J.L.: Spectral distribution of typical daylight as a function of correlated color temperature. J. Opt. Soc. Am. **54**(8), 1031–1040 (1964)

[15] Kobayashi, S., Nomizu, K.: Foundations of differential geometry. Vol. I. Wiley Classics Library. John Wiley & Sons, Inc., New York (1996). Reprint of the 1963 original, A Wiley-Interscience Publication

[16] Kurdgelaidze, D.F., Khukhunashvili, Z.V.: Determination of the Lie groups of equations of motion. i. Soviet Physics Journal **11**(10), 35–40 (1968). DOI 10.1007/BF00822464

[17] Mahony, R., Manton, J.H.: The geometry of the newton method on non-compact lie groups. Journal of Global Optimization **23**(3), 309–327 (2002). DOI 10.1023/A:1016586831090

[18] Mahony, R.E.: The constrained Newton method on a Lie group and the symmetric eigenvalue problem. Linear Algebra Appl. **248**, 67–89 (1996). DOI 10.1016/0024-3795(95)00171-9

[19] Munthe-Kaas, H.Z., Zanna, A.: Numerical integration of differential equations on homogeneous manifolds. In: F. Cucker, M. Shub (eds.) Foundations of Computational Mathematics, pp. 305–315. Springer Berlin Heidelberg, Berlin, Heidelberg (1997)

[20] Owren, B., Welfert, B.: The Newton iteration on Lie groups. BIT **40**(1), 121–145 (2000). DOI 10.1023/A:1022322503301

[21] Rivertz, H.J.: An obstruction to isometric immersions of the three dimensional Heisenberg group into $\mathbf{R}^4$. Preprint series Institute of Mathematics, University of Oslo. Pure mathematics 22 (1999)

[22] Rivertz, H.J.: On isometric and conformal immersions into Riemannian manifolds. Doctor thesis in mathematics, University of Oslo, Department of mathematics, Mathematics division (1999)

[23] Rivertz, H.J.: On an invariant on isometric immersions into spaces of constant sectional curvature. Ukr. Mat. Zh. **61**(12), 1660–1667 (2009). DOI 10.1007/s11253-010-0324-8

[24] Smith, T., Guild, J.: The c.i.e. colorimetric standards and their use. Transactions of the Optical Society **33**(3), 73

[25] Varadarajan, V.S.: Lie groups, Lie algebras, and their representations, *Graduate Texts in Mathematics*, vol. 102. Springer-Verlag, New York (1984). DOI 10.1007/978-1-4612-1126-6. Reprint of the 1974 edition

[26] Vora, P.L., Trussell, H.J.: Measure of goodness of a set of color-scanning filters. J. Opt. Soc. Am. A **10**(7), 1499–1508 (1993)

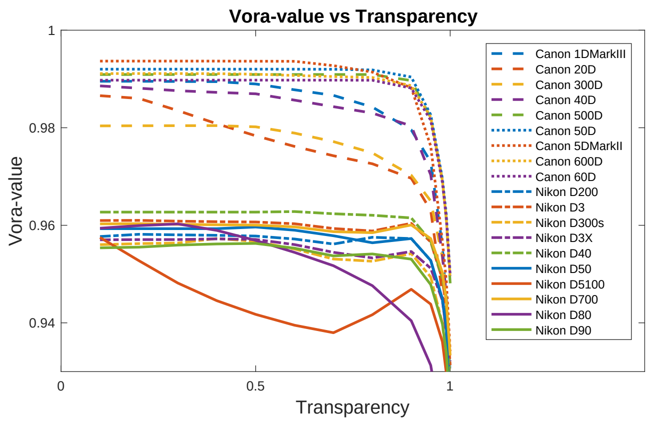


Fig. 5. The Vora-value against the luminance transparency of daylight for the cameras in consideration.