https://doi.org/10.2352/issn.2169-2629.2019.27.18

On an Euler-Lagrange equation for color to grayscale conversion

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Abstract

In this paper we give a new method to find a grayscale image from a color image. The idea is that the structure tensors of the grayscale image and the color image should be as equal as possible. This is measured by the energy of the tensor differences. We deduce an Euler-Lagrange equation and a second variational inequality. The second variational inequality is remarkably simple in its form. Our equation does not involve several steps, such as finding a gradient first and then integrating it. We show that if a color image is at least two times continuous differentiable, the resulting grayscale image is not necessarily two times continuous differentiable.

Introduction

Scope of this paper

Our paper presents a new method and framework for color to grayscale conversions. Our method has made it possible to study the core of one of the hardest problems in image science, i.e. the non integrability problems that occur in color to grayscale.

Our method is based on variational calculus, which is a mathematical tool used in physics but also in differential geometry and other branches of mathematics. We apply variational calculus as in Lagrange formalism of classical mechanics to prove a fundamental differential equation for color to grayscale conversion. This is a so called Euler-Lagrange equation. We will also go one step further and find the so called second variational formula for the conversion problem. This is rarely done in image processing. A solution of the Euler-Lagrange equation gives only critical point. The second variational formula tells us when a critical point gives a minimum solution.

A similar but simpler approach was followed in [3] by Ali Alsam and the author. Instead of using a variational approach, we searched for luminance maps that preserved as much as possible of the contrast. The idea to compute and investigate the second variational formula was inspired by works in differential geometry [6], [12].

On gradient methods

In color to grayscale transform of a picture, finding the gradient is often the first task to be solved, ([1], [2], [8], [10]). The gradient is also used in other applications, such as edge detection, ([5], [15], [13], [7], [13], [9]), and image fusion, ([14], [11], [4]). Many methods for finding a gradient are known. The most celebrated method is to use an eigenvector of the structure tensor. We take an eigenvector belonging to the highest eigenvalue. The length is set to the square root of the eigenvalue, [16]. This method is used in most of the articles cited in the present article. The method has two properties that are worth mentioning.

P1 For a grayscale image, the method will give the picture's exact gradient up to sign. This fact is often used to justify the eigenvector method.

P2 A pleasant fact about the method is that the eigenvalues and the eigenspaces are invariant under orthonormal coordinate transformations.

In most gradient based methods in image analysis, if (p,q) is a candidate for the gradient of a picture described by a function f, (p,q) is generally not integrable, that is, it is not a gradient field.

One way to find the grayscale image is to solve the PDE

$$p_x + q_y = \Delta f.$$

In this article we give a PDE in the unknown f that gives a direct way to find f from the structure tensor. We also show a condition where the eigenvalue/vector method gives exactly the same solution as our method.

From color gradient to grayscale gradient

Model of the image

We view an n-channel multi spectral image as a smooth function $\mathbf{f}(x,y)$. Its domain is the "canvas" $\Omega = I_W \times I_H$, where I_W and I_H are intervals of the real line. The function $\mathbf{f}$ takes its values in a color space (C,g). C can be viewed, at least locally, as

$$\mathbb{R}^n = \{(y_1, y_2, \ldots, y_n) | y_i \in \mathbb{R}\}$$

with a metric $g = [g_{ij}]$. This metric defines the curve length differential ds by

$$ds^2 = \sum_{i,j} g_{ij} dy_i dy_j. \tag{1}$$

The metric coefficients are symmetric, e.g. $g_{ij} = g_{ji}$. In general, the coefficients g_{ij} are functions on C.

The structure tensor

The color gradient is the ordered pair of vectors $(\mathbf{f}_x, \mathbf{f}_y)$,

$$\mathbf{f}_x = \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}, \ldots, \frac{\partial f_n}{\partial x} \right) \tag{2}$$

and

$$\mathbf{f}_y = \left(\frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial y}, \ldots, \frac{\partial f_n}{\partial y} \right). \tag{3}$$

The color structure tensor is the matrix field

$$M_C = \begin{bmatrix} g(\mathbf{f}_x, \mathbf{f}_x) & g(\mathbf{f}_x, \mathbf{f}_y) \\ g(\mathbf{f}_y, \mathbf{f}_x) & g(\mathbf{f}_y, \mathbf{f}_y) \end{bmatrix}, \tag{4}$$

where $g(X,Y) = \sum_{ij} g_{ij} X_i Y_j$. In the case of grayscale images, the "color" function is a real function $f(x,y)$. We then have

$$M_G = \begin{bmatrix} {f_x}^2 & f_y f_x \\ f_y f_x & {f_y}^2 \end{bmatrix}. \tag{5}$$

We call this the gradient tensor field. The gradient tensor field has the following properties. The determinant is equal to 0 and it has two eigenvalues $\lambda = {f_x}^2 + {f_y}^2$ and 0. The gradient $f_x\mathbf{i} + f_y\mathbf{j}$ is an eigenvector for the eigenvalue λ. The square root $\sqrt{\lambda}$ of the eigenvalue is the length of the gradient. Another property of the gradient is the integrability condition for gradient vectors; $f_{xy} = f_{yx}$. This is actually the mixed derivative theorem and requires that f has continuous second derivatives.

Given a structure tensor M, e.g. M_C, over Ω. We will search for the least square approximation of

$$M_C = \begin{bmatrix} A & C \\ C & B \end{bmatrix}$$

in the space of grayscale structure tensors, i.e. matrix fields on the form

$$\begin{bmatrix} X & Z \\ Z & Y \end{bmatrix}$$

where $X = f_x^2$, $Y = f_y^2$ and $Z = f_x f_y$.

The functional

$$W(f) = \iint_\Omega \Big[(X-A)^2 + 2(Z-C)^2 + (Y-B)^2 \Big] dx\, dy \quad (6)$$

is then minimized. We will assume that there exists a solution f to this minimizing problem with continuous partial derivatives of second order. In the section named *Picture in trouble*, we will in fact show that this is not always true. The factor 2 in the second term in the integrand is necessary for making W invariant with respect to O.N. changes of coordinates of Ω.

Variation approach

The Euler-Lagrange equation of the color to grayscale problem

In this section we will prove an Euler-Lagrange equation for the color to grayscale problem. That is a differential equation that is satisfied for minimal solutions f for the functional $W(f)$.

Theorem 1. *If $f : \Omega \to \mathbb{R}$ is a grayscale image that minimizes the integral W in Equation* (6) *and f has continuous partial derivatives of second order, then f satisfies*

$$\frac{\partial}{\partial x}\Big((X+Y)\, f_x - A\, f_x - C\, f_y \Big) + \frac{\partial}{\partial y}\Big((X+Y)\, f_y - C\, f_x - B\, f_y \Big) = 0. \quad (7)$$

Proof. Let $X = f_x^2$, $Y = f_y^2$, and $Z = f_x f_y$ minimize W and let η be a smooth function defined on Ω, so that η is zero on the boundary of Ω. Consider the variation $\tilde{f}(t) = f + t\eta$. Then, $\tilde{X}(t) = X + 2f_x\eta_x\, t + \eta_x^2\, t^2$, $\tilde{Y}(t) = Y + 2f_y\eta_y\, t + \eta_y^2\, t^2$, and $\tilde{Z}(t) = Z + (f_x\eta_y + f_y\eta_x)\, t + \eta_x\eta_y\, t^2$. Let $F(t) = (\tilde{X}(t) - A)^2 + 2(\tilde{Z}(t) - C)^2 + (\tilde{Y}(t) - B)^2$. The derivative of $F(t)$ is

$$\begin{aligned}\frac{d}{dt}F(t) &= 4(\tilde{X}(t) - A)\cdot(f_x\eta_x + \eta_x^2\, t) \\ &+ 4(\tilde{Z}(t) - C)\cdot(f_x\eta_y + f_y\eta_x + 2\eta_x\eta_y\, t) \\ &+ 4(\tilde{Y}(t) - B)\cdot(f_y\eta_y + \eta_y^2\, t). \quad (8)\end{aligned}$$

For $t = 0$, this is

$$\begin{aligned}\frac{d}{dt}F(0) &= 4\Big((X-A)\, f_x + (Z-C)\, f_y \Big)\eta_x \\ &+ 4\Big((Z-C)\, f_x + (Y-B)\, f_y \Big)\eta_y \\ &= 4\Big((X+Y)\, f_x - A\, f_x - C\, f_y \Big)\eta_x \\ &+ 4\Big((X+Y)\, f_y - C\, f_x - B\, f_y \Big)\eta_y. \quad (9)\end{aligned}$$

We have used that $Y f_x = Z f_y$ and that $X f_y = Z f_x$.

$$\begin{aligned} 0 &= \frac{d}{dt}W(f_t)|_{t=0} = \iint_\Omega \frac{d}{dt}F(0)\, dx\, dy \\ &= 4\iint_\Omega \Big((X+Y)\, f_x - A\, f_x - C\, f_y \Big)\eta_x \\ &+ \Big((X+Y)\, f_y - C\, f_x - B\, f_y \Big)\eta_y\, dx\, dy \\ &= 4\iint_\Omega \frac{\partial}{\partial x}\Big((X+Y)\, f_x - A\, f_x - C\, f_y \Big)\eta\, dx\, dy \\ &+ 4\iint_\Omega \frac{\partial}{\partial y}\Big((X+Y)\, f_y - C\, f_x - B\, f_y \Big)\eta\, dx\, dy. \end{aligned}$$

Partial integration is used to establish the last equality. Given an interior point (x_0, y_0) of Ω and a positive real number s so that the disc $D_s(x_0, y_0) = \{(x,y) | (x-x_0)^2 + (y-y_0)^2 \le s^2\}$ is contained in the interior of Ω. First define $N(x,y) = (x-x_0)^2 + (y-y_0)^2 - s^2$. Let $\eta(x,y)$ be given as $c\exp\left(s^2/N(x,y)\right)$ on the interior of $D_s(x_0, y_0)$ and zero everywhere else. The real number c is a constant so that $\iint_\Omega \eta\, dx\, dy = 1$. Since η is non-negative on D_s and the left hand side of equation (7) is continuous on Ω there must be a point in D_s where equation (7) is satisfied. Therefore, since (x_0, y_0) and s are chosen arbitrarily, equation (7) is satisfied on a dense subset of Ω and hence it is satisfied everywhere on Ω. □

The second variational formula for the color to grayscale problem

In this subsection, we give an inequality that must be satisfied if any solution f of the Euler-Lagrange equation is a minimum for the functional $W(f)$.

Theorem 2. *If $f : \Omega \to \mathbb{R}$ is a function with continuous second derivatives that minimizes the integral W in Equation* (6)*, then f satisfies*

$$|\nabla f|^2 \ge \frac{1}{4}(A+B) \quad (10)$$

at every point in Ω.

Proof. We calculate

$$0 \le \frac{d^2}{dt^2}W(f(t))|_{t=0} = \iint \frac{d^2}{dt^2}F(0)\, dx\, dy.$$

The second derivative of $F(t)$ at $t = 0$ is

$$\begin{aligned}\frac{d^2}{dt^2}F(0) &= 4(3X + Y - A)\cdot\eta_x^2 \\ &+ 8(2Z - C)\cdot\eta_x\eta_y + 4(3Y + X - B)\cdot\eta_y^2. \quad (11)\end{aligned}$$

Let $\eta(x,y) = c\exp\left(s^2/N\right)$ for $(x-x_0)^2 + (y-y_0)^2 < s^2$ and $\eta(x,y) = 0$ elsewhere, where $N = (x-x_0)^2 + (y-y_0)^2 - s^2$. We restrict s to values such that the support of $\eta(x,y)$ is contained in Ω. The first derivatives of η are

$$\begin{aligned}\eta_x &= -\frac{2(x-x_0)s^2}{N^2}\eta \\ \eta_y &= -\frac{2(y-y_0)s^2}{N^2}\eta.\end{aligned}$$

In s-independent polar coordinates centered around the point (x_0, y_0) given by $x = x_0 + sr\cos\theta$ and $y = y_0 + sr\sin\theta$, the above formulas are

$$\eta_x = -\frac{2cr\cos\theta}{s(r^2-1)^2}\exp\left(\frac{1}{r^2-1}\right)$$
$$\eta_y = -\frac{2cr\sin\theta}{s(r^2-1)^2}\exp\left(\frac{1}{r^2-1}\right).$$

The integral of η_x^2 and η_y^2 over Ω is $4\pi c^2\int_0^1 \frac{r^3}{(r^2-1)^4}e^{\frac{2}{r^2-1}}dr$, while the integral of $\eta_x\eta_y$ in the first sector $(0\le\theta\le\pi/2)$ is $2c^2\int_0^1\frac{r^3}{(r^2-1)^4}e^{\frac{2}{r^2-1}}dr$. Since η_x^2 and η_y^2 are positive on their support, the lemma follows. The term with $\eta_x\eta_y$ will vanish under integration when s goes to 0. We have used that A, B, etc. are continuous. □

The boundary condition

We assume a reflexive boundary condition for the images. A matrix version of equation (7) is

$$\begin{bmatrix} A-|\nabla f|^2 & C \\ C & B-|\nabla f|^2\end{bmatrix}\begin{bmatrix} f_x \\ f_y\end{bmatrix} = \begin{bmatrix} -K_y \\ K_x\end{bmatrix}, \tag{12}$$

where $K(x,y)$ is a two times differentiable function on Ω. The function $K(x,y)$ exists if $(A-|\nabla f|^2)f_x + Cf_y$ and $Cf_x+(B-|\nabla f|^2)f_y$ have continuous partial derivatives. The value of K is constant on the boundary of Ω: On the vertical boundaries we have $X=A=C=0$ and $f_x=0$. Thus, we have

$$\begin{bmatrix} -Y & 0 \\ 0 & B-Y\end{bmatrix}\begin{bmatrix} 0 \\ f_y\end{bmatrix} = \begin{bmatrix} 0 \\ (B-Y)f_y\end{bmatrix} = \begin{bmatrix} -K_y \\ K_x\end{bmatrix}, \tag{13}$$

so $K_y=0$ and therefore K is constant on the left and right vertical boundaries. The same argument holds for the horizontal boundaries.

Lemma 1. *Assume that $f:\Omega\to\mathbb{R}$ is a grayscale image that minimizes the integral W in (6) and f has continuous partial derivatives of second order. The values of X and Y on the boundary are then equal to A and B respectively.*

Proof. $((B-Y)f_y)_y = 0$ on the vertical boundaries. This means that K_x is constant on the boundary. Since K_x is zero in the corners, then $K_x=0$ along the boundary. Thus, $(B-Y)f_y=0$. By the second variational formula, we have $5Y\ge B$ on the vertical boundaries. Thus, $Y=B$. A similar argument gives $X=A$ on the horizontal boundaries. □

Corollary 1. $\nabla K = \mathbf{0}$ *on the boundary.*

A remark on the eigenvalue method

In the previous section, we saw that $K(x,y)$ is constant on the boundary. On Ω, $K(x,y)$ is generally not a constant function, but in the case where K is a constant function, we have an eigenvalue problem. In fact, let λ' be the highest eigenvalue of M_C with corresponding eigenvector (P,Q) with length $\sqrt{\lambda'}$. In the mainstream literature (P,Q) is used as a non-integrable grayscale gradient. That is $X'=P^2$, $Y'=Q^2$, $Z'=PQ$, $\lambda'=X'+Y'$, $AP+CQ=\lambda'P$, and $CP+BQ=\lambda'Q$. Combining these equations gives $AP+CQ=(X'+Y')P$ and $CP+BQ=(X'+Y')Q$. Therefore the equation in the theorem is "satisfied" by replacing f_x and f_y with P and Q respectively. A non vanishing K says that the eigenvector field is not integrable. The highest eigenvalue gives

$$|\nabla f|^2 = \lambda' = \frac{A+B+\sqrt{(A+B)^2-4AB+4C^2}}{2} \ge \frac{1}{4}(A+B).$$

Thus, theorem 2 proves that the highest eigenvalue method minimizes $W(f)$ when (P,Q) is integrable.

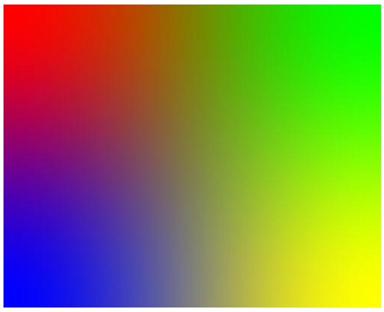

Figure 1. *Synthetic image in trouble.*

Picture in trouble

In this section we consider the synthetic image defined by the RGB-function $\mathbf{f}(x,y) = [1-S(x)-S(y)+2S(x)S(y), S(x), (1-S(x))S(y)]$, where $S(t)=(3-2t)t^2$. The canvas is $[0,1]\times[0,1]$. The image is displayed in Figure 1. The color gradient has components $\mathbf{f}_x(x,y) = [-S'(x)+2S'(x)S(y), S'(x), -S'(x)S(y)]$ and $\mathbf{f}_y(x,y) = [-S'(y)+2S(x)S'(y), 0, (1-S(x))S'(y)]$. Integration of the direction derivative of f along the edge should give 0, but we get $\pm\sqrt{2}\pm\sqrt{3}\pm\sqrt{1}\pm\sqrt{2}\neq 0$. Therefore, given the reflection boundary condition, a minimal solution of the energy equation (6) either does not exist for this special image or it does not have continuous second derivatives. Another metric g for calculating M_C could fix the problem, but it would only be an ad hoc solution.

Upper edge	$\mathbf{f}_x(x,0) = [-S'(x), S'(x), 0]$ $\mathbf{f}_y(x,0) = [0,0,0]$ $X=A=2S'(x)^2$
Lower edge	$\mathbf{f}_x(x,1) = [S'(x), S'(x), -S'(x)]$ $\mathbf{f}_y(x,1) = [0,0,0]$ $X=A=3S'(x)^2$
Left edge	$\mathbf{f}_x(0,y) = [0,0,0]$ $\mathbf{f}_y(0,y) = [-S'(y), 0, S'(y)]$ $Y=B=2S'(y)^2$
Right edge	$\mathbf{f}_x(1,y) = [0,0,0]$ $\mathbf{f}_y(1,y) = [S'(y),0,0]$ $Y=B=S'(y)^2$

The color gradients along the edges of the image in trouble.

Conclusion

In this paper we propose a new method to study the process of color to grayscale conversion. We introduce an energy functional $W(f)$ for the grayscale structure tensor, which give us an Euler-Lagrange equation for color to grayscale conversion.

Our method improves the eigenvector method for estimating a grayscale gradient field from a color image in the sense that our method does not need to find a direction of tangent fields. The picture used in our example shows that one should be careful when assumptions are laid on the grayscale solution. A better approach to color to grayscale conversion and other image fusion methods, could be to look for solutions with discontinuous partial derivatives or second partial derivatives.

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