Metamer mismatch volumes using spherical sampling

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Abstract

In this article, we propose a fast and precise method of calculating a theoretically maximum metamer mismatch volume. Unlike earlier proposed techniques, our method does not use any assumptions on the shape of spectra on the boundary of the mismatch volume. The method utilises a spherical sampling approach and a linear programming optimisation. Experiments described in this paper show that under certain conditions the theoretically maximum metamer mismatch volume is significantly larger than the one approximated using previous method that makes assumptions about the shape of the spectra.

Introduction

Metamer mismatiching refers to the phenomenon whereby two objects modelled by their surface reflectance spectra appear to an observer the same colour under one light and different under another light [1]. Analogously, when there are two observers, two objects may appear the same colour to one and different colours to another. This raises the question about the extent of this phenomenom i.e. given the observed colour match under the first condition, what is the range of possible colours that the two objects may take under the second condition. The set of all such colours is called the metamer mismatch volume.

Metamer mismatch volumes are relevant from the point of view camera sensor design. We do not wish to manufacture colour sensors producing large mismatch volumes for the change of observer from camera to a human. Yet, in practice modern cameras do not meet this requirment and hence some level of metamer mismatching is unavoidable. Consequently, understanding metamer mismatching becomes important from the point of view of developing colour correction algorithms. Another important application where metamerism plays a significant role is lighting design. Of course, understanding mismatch volumes also teaches us about our own vision (it is an interesting question in its own right).

The attempts to estimate the size of metamer mismatch volumes were initially underpinned by the assumption that all object spectra can be modelled by a low dimensional linear model [2, 3, 4]. More recently, Logvinenko et al. in [5] proposed to calculate the full extent of the metamer mismatch volume. The approach they have chosen for describing these objects was to calculate their precise boundaries. The rationale for this approach stems from the observation that there is no metamerism at the boundaries of these volumes i.e. a colour response at the boundary of the mismatch volume maps a unique spectral reflectance function. Then, a question may be posed: what is the general form of the shape of these boundary reflectance spectra? Logvinenko et al. noted that these are elementary step functions of zeros and ones with some limited number of transitions between these two values. They then parametrised these functions with respect to the wavelengths where the transitions between 0 and 1 occur and conjectured that the boundary reflectance function model with no more than five transitions should provide a good approximation of the theoretically maximum volumes. However, as we will later show in our experiments, their chosen parametrisation results in significant underestimation of the theoretically maximum volumes of the mismatch objects.

The rest of the paper is organised as follow. First, we introduce the relevant theory of mismatch volumes and notation and briefly describe the Logvinenko et al.'s algorithm [5]. Then, we describe our approach. Finally, we describe experiments evaluating the effectiveness of our algorithm and compare it to the prior art.

Metamer Mismatch Volumes Theory

The theory of metamer mismatching has been described by Logvinenko et al. in [5]. Below, we recapitulate their major points. We adopt their mathematical notation.

The colour responses of the set of N sensors $\Phi(r) = (\phi_1(r), \phi_2(r), ..., \phi_N(r))$ to an object with a spectral reflectance function $r(\lambda)$ illuminated by the light with the spectral power distribution $e(\lambda)$ are given by the following equation, which is often referred to as the colour formation equation:

$$\phi_i(r) = \int_{\lambda_{min}}^{\lambda_{max}} r(\lambda)e(\lambda)c_i(\lambda)\,\mathrm{d}\lambda, \quad i = 1, 2, .., N. \tag{1}$$

where λ_{min} and λ_{max} denote the limits of the visible spectrum and c_i are the sensor spectral sensitivities. A reflectance spectrum $r(\lambda)$ is a function with values between zero and one.

In case of human vision, N would be 3 and the the sensor sensitivities $c_i(\lambda)$ could be e.g cone fundamentals or colour matching functions [1]. Attempting to emulate the colorimetry of the tristimulus human visual system, most modern digital cameras would also employ three spectral sensitivity functions.

The two objects with reflectance functions $r(\lambda)$ and $r'(\lambda)$ are called metameric if they produce the same colour signals $\Phi(r) = \Phi(r')$. It is clear from Eq. 1 that two objects may cease to be metameric if either illuminaton or the sensor spectral sensitivities change resulting in *illumination-induced* or *observer-induced metamer mismatching* respectively. More precisely, Eq. 1 tells us that metamer mismatching is affected by the product of the illuminant spectrum and the sensor sensitivities, which we will refer to as the *colour sytem* spectra i.e. $\mathbf{s}(\lambda) = \mathbf{c}(\lambda)e(\lambda)$. The above can be substituted into Eq. 1 resulting in:

$$\Phi(r) = \int_{\lambda_{min}}^{\lambda_{max}} r(\lambda)\mathbf{s}(\lambda)\,\mathrm{d}\lambda. \tag{2}$$

We will consider another set of colour responses $\Psi = (\psi_1, .., \psi_n)$ corresponding to the colour system spectra $\mathbf{s}'(\lambda) =$

$(s'_1(\lambda),...,s'_n(\lambda))$. The new colour system might have resulted from the alteration of the illuminant spectrum, sensor sensitivities or in general both. Logvinenko et al. [5] point out that both Φ and Ψ can be considered as linear maps: $\mathscr{X} \to \mathbb{R}^N$, where $\mathscr{X}$ denotes the set of all reflectance functions. The sets of all possible colour system responses of both Φ and Ψ form convex sets in $\mathbb{R}^N$, which are called *object colour solids* (OCS).

The *metamer set* with respect to the colour system Φ can be defined as the set of all reflectances metameric to a given reflectance r_0 i.e. $\Phi^{-1}(\Phi(r_0)) = \{r \in \mathscr{X} | \Phi(r) = \Phi(r_0)\}$. In general, the above metamer set will be mapped by Ψ to a non-singleton set which is referred to as the *metamer mismatch volume*.

Logvinenko et al. [5] introduced an additional linear map $\Gamma : \mathscr{X} \to \mathbb{R}^{2N}$ such that $\Gamma(r) = (\mathbf{z}, \mathbf{z}')$, where $\mathbf{z} = \Phi(r)$ and $\mathbf{z}' = \Psi(r)$. Consequently, $\Gamma(\mathscr{X})$ is the object colour solid in $\mathbb{R}^{2N}$. The authors observed that for the colour response $\Phi(r_0) = \mathbf{z}_0$, the metamer mismatch volume $\mathscr{M}(\mathbf{z}_0, \Phi, \Psi) = \Psi(\Phi^{-1}(\mathbf{z}_0) = \{\mathbf{z}' \in \mathbb{R}^N : (\mathbf{z}_0, \mathbf{z}') \in \Gamma(\mathscr{X})\}$ is a cross-section of $\Gamma(\mathscr{X})$.

The OCS can be described by its boundary. The attractiveness of such a description stems from the fact that unlike the points in the interior of the OCS that represent different *metameric classes* where each class can be mapped to a metamer set comprising infinitely many spectra, each point on the OCS boundary has only one corresponding reflectance spectrum. These 'boundary' spectra are called *optimal* and have the property of being elementary step functions of zeros and ones.

It was proposed by Shrödinger that the elementary step functions on the surface of the OCS created from the human sensors do not have more than two transitions (we call these - $m < 3$ spectra) [6]. This was later shown to be incorrect [7, 8]. This said, the approximation of the human vision OCS with the ($m < 3$) spectra has been found to be relatively accurate [8].

The mathematical description of the optimal spectra on the boundary of the OCS has been given in [8]. The authors proposed that for the set of N colour systems the spectra located on the boundary of the object colour solid (*optimal spectra*) are the elementary step functions with the transition wavelengths at $\lambda_1, ..., \lambda_m$ if and only if the above set of transition wavelengths are the only zero-crossings of the following equation:

$$k_1 s_1(\lambda) + k_2 s_2(\lambda) + ... + k_N s_N(\lambda) = 0, \tag{3}$$

where $k_1, k_2, ..., k_N$ are the set of arbitrary real numbers, where at least one of them is not equal to zero.

Logvinenko et al. [5] also proposed to describe the metamer mismatch volume denoted as $\mathscr{M}(\mathbf{z}_0, \Phi, \Psi)$ with its boundary $\partial\mathscr{M}$ and called the 'boundary' spectra $\mu-$*optimal*. They admit that the optimal spectra on the boundary of Γ and the μ-optimal spectra on the boundary of $\mathscr{M}$ can have an arbitrary number of transitions as determined by Eq. 3. This said, they chose to approximate the boundaries of the above volumes using a parametrisation of their corresponding optimal or μ-optimal spectra constraining them to the elementary step functions with up to five transitions ($m < 6$). This choice was motivated by the Schrödinger's conjecture regarding the elementary functions on the 3-D human OCS. Analogously to the 3-D OCS boundary conjectured to be descibed with the optimal spectra with up to $3 - 1 = 2$ transitions, they proposed to approximate the boundary of the 6-D Γ and its cross-section $\mathscr{M}$ with the elementary spectra with up to $6 - 1 = 5$ transitions. Following the aforementioned authors, we will denote the above two boundary approximations as $\partial\Gamma(\mathscr{O}_5)$ and $\partial\mathscr{M}_5$ respectively.

The idea of describing μ-optimal spectra with elementary step functions with up to five transtions is not new. Ohta and Wyszecki held the (incorrect) view that the metamer mismatch volume boundary spectra are precisely such spectra [9]. Logvinenko et al. recognised that this is not the case and admitted that a model imposing such a restriction can describe the boundary of these objects only approximately. Nevertheless, they adopted this model in their algorithm which we summarise below.

Calculating the boundary of the mismatch volume in practice

The following algorithm for calculation of $\partial\mathscr{M}_5(\mathbf{z}_0, \Phi, \Psi)$ was proposed in [5]. The algorithm begins by generating a large number of optimal spectra in $\partial\Gamma(\mathscr{O}_5)$ i.e $m < 6$ spectra. Then, for each generated spectrum denoted as $r_{opt} = r_5(\lambda_1, ..., \lambda_5)$ it performs the following optimisation:

$$\min_{r_{opt}} ||\Phi(r_{opt}) - \mathbf{z}_0||. \tag{4}$$

In the next section we will briefly describe our recently published algorithm for calculation of the OCS boundary comprising optimal spectra with any number of transitions. This will lay the foundation of the main contribution of this paper that is an algorithm for calculation of the boundary of the mismatch volume not limited by the number of transitions of μ-optimal spectra.

Calculating the boundary of the OCS using spherical sampling

In [10], we observed that the components of vector $\mathbf{k}$ in Eq. 3 have the geometrical meaning i.e. they constitute the normal vector parametrising the boundary of the object colour solid. Because the OCS is convex, in the direction $\mathbf{k}$, we can, in closed form, find the unique system response which is maximum. As the OCS is convex it follows that we can find all points on the OCS by maximising or minimising all directions. Formally, we propose the parametric representation with respect to $\mathbf{k}$ of the boundary of the object colour solid.

We project all colour system responses $\Phi(r) = (\phi_1(r), \phi_2(r), ..., \phi_N(r))$ from Eq. 2 onto a unit vector $\mathbf{k}$. That is

$$\mathbf{k} \cdot \Phi(r) = \int_{\lambda_{min}}^{\lambda_{max}} r(\lambda)\, \mathbf{k} \cdot \mathbf{s}(\lambda)\, d\lambda$$

.

It is clear that the maximum value of $\mathbf{k} \cdot \Phi(r)$ is obtained by

$$r_{opt} = r(\lambda; \mathbf{k}) = \begin{cases} 0, & \mathbf{k} \cdot \mathbf{s}(\lambda) < 0 \\ 1, & \mathbf{k} \cdot \mathbf{s}(\lambda) \geq 0 \end{cases}. \tag{5}$$

The above observation leads to a very rapid algorithm for calculation of the optimal specra on the boundary of the OCS.

We can generate a set of M normal vectors in R^N and store them in the rows of $M \times N$ matrix $\mathbf{P}$ [11]. We store colour system spectra in $N \times q$ matrix $\mathbf{S}$. The wavelength resolution is determined by q e.g. for 1nm resolution, $\lambda_{min} = 380$ and $\lambda_{max} = 730$, the colour system and reflectance spectra will have 351 components i.e. $q = 351$.

We denote a matrix resulting from multiplication of $\mathbf{P}$ by $\mathbf{S}$ as $\mathbf{A} = \mathbf{PS}$. Then, the signs of the elements of $\mathbf{A}$ determine the set of optimal spectra in matrix $\mathbf{R}$ as:

$$\mathbf{R}_{ij} = \begin{cases} 0, & \mathbf{A}_{ij} < 0 \\ 1, & \mathbf{A}_{ij} \geq 0 \end{cases} \tag{6}$$

The algoirthm requires no search and hence it is very fast.

Calculation of the OCS is not a significant problem and there were a number of algorithms proposed in the past. For example, for the set of three sensors, one could generate a number of sensor responses from the set of randomly generated elementary spectra with two transitions [12, 13]. This said, our method allows for describing the OCS with a small number of samples and it also allows optimal spectra with any high number of transitions. Further, it naturally leads to a related algorithm for calculation of the boundary of the metamer mismatch volume which will be introduced in the next section.

Calculating the boundary of the metamer mismatch volume using spherical sampling

The algorithm presented in the previous section allows for the calculation of either $\partial\Phi \in \mathbb{R}^N$ or the larger $\partial\Gamma \in \mathbb{R}^{2N}$. Earlier we noted that the metamer mismatch volume $\mathscr{M}$ is a cross section of Γ and it is convex. Then, analogously to the algorithm presented in the previous section we can find $\partial\mathscr{M}(\mathbf{z}_0, \Phi, \Psi)$ by extremizing all spherically sampled directions $\mathbf{k}$ in $\mathbf{R}^{2N}$ subject to $\Phi(r) = \mathbf{z}_0$ and further constraints on the values of the reflectance function. Thus, we can write the above as the following optimisation:

$$\max_r \int_{\lambda_{min}}^{\lambda_{max}} r(\lambda)\mathbf{k}\cdot\mathbf{s}(\lambda), \tag{7}$$

subject to

$$\Phi(r) = \mathbf{z}_0$$

$$0 < r(\lambda) < 1,$$

where $\mathbf{s}(\lambda)$ are the $2N$ colour system spectra.

Next, we choose a wavelength sampling resolution and write the above optimisation using vector notation as a linear programmng problem:

$$\max_{\mathbf{r}} (\mathbf{Sk})^T \mathbf{r}, \tag{8}$$

subject to

$$\mathbf{S}_\Phi^T \mathbf{r} = \mathbf{z}_0$$

$$0 < \mathbf{r}_i < 1 \quad \text{for} \quad i = 1, \ldots, q,$$

where $\mathbf{S}$ is a $q \times 2N$ matrix containing colour system spectra, $\mathbf{S}_\Phi$ is a $q \times N$ matrix containing colour system spectra of Φ and $\mathbf{r}$ is a q-vector containing a μ-optimal spectrum in $\partial\mathscr{M}(\mathbf{z}_0, \Phi, \Psi)$ that corresponds to the direction $\mathbf{k}$.

Additional improvement can be achieved by using the orthornormal set of colour system spectra which has the pottential of achieving a more uniform sampling of the boundaries of both OCS and the metamer mismatch volume. Then, optimisation can be written as:

$$\max_{\mathbf{r}} (\mathbf{Uk})^T \mathbf{r}, \tag{9}$$

subject to the same constraints as (8), where $\mathbf{U}$ is a $q \times 2N$ matrix containing the set of orthonormal colour system spectra which can be obtained from $\mathbf{S}$ using singular value decomposition [14].

Experiments and Results

We performed a number of experiments testing the two variants of the proposed algorithm and comparing them with the algorithm presented in [5]. In all experiments we used 1931 colour matching functions. We built metamer mismatch volumes for the change of illuminant for three illuminants: D65, A and F11 [15] for all 6 pairs. We will show only the results of the two conditions for the change of illuminant from D65 to A and F11 to D65, which are representative of all the results.

In our first experiment, we wished to determine the appriopriate wavelength sampling for our algorithm. We tested wavelength resolutions from 0.2nm to 10nm. The metamer mismatch volumes were calculated for the flat grey reflectance ($r_0 = 0.5$) for the increasing number of samples generated using spherical sampling as described in the earlier section. Here, we used the second variant of our algorithm i.e. the orthonormal colour system spectra. We also compared the volumes generated with this algorithm with those obtained using [5]. The results of these experiments are shown in Figures 1 and 2.

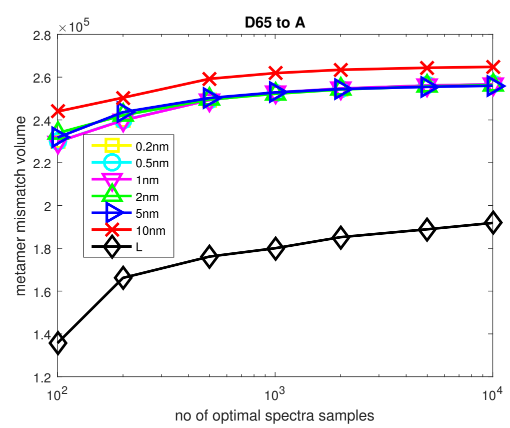

Figure 1. *Metamer mismatch volumes calculated for the flat grey reflectances with 50% reflectivity for the two sets of orthormal colour sytem specra for the change of illuminant from D65 to A with spectral sampling varying from 0.2nm to 10nm. (L) - method proposed by Logvinenko et al. [5]. Results for sampling resolutions from 0.2nm to 1nm are almost the same and hence are hidden under the 1nm plot.*

In Fig. 1, we can see that the volumes obtained for wawelength sampling resolutions from 0.2nm to 1nm are almost identical. The 2nm and and 5nm are very close and the 10nm sampling overestimates the volume. These observations are confirmed in Fig. 2, where again we can see that sampling resolutions from 0.2 to 1nm produce the same result. The errors for higher wavelength resolutions become more visible as this result has been produced for the change of illuminant from F11, which has a spectrum that is less smooth than both D65 and A. Therefore, for all subsequent experiments we chose the wavelength resolution of 1nm.

Figures 1 and 2 show that the volumes obtained for the prior art method are much smaller (from approximately 25% to 70%), particularly for a small number of samples and when the F11 illuminant is involved.

Figures 3 and 4 show the results of the second experiment where we investigated two further aspects of our algoirhtm. First, we look at the volumes produced by the two versions of the algorithm i.e. with and without the orthonormal colour system spectra and second, the size of the mismatch volumes along the achromatic line. It can be seen that the use of orthornal colour system spectra indeed results in better distribution of the samples describing the mismatch volume. More specifically, in Fig. 4 we can see that for the flat grey reflectance (50%), the version of the algoirthm utilisng the orthonormal spectra with 100 spherical samples produces the mismatch volume estimate that is matched only by the original version of the algorithm with 500 samples.

As to the second aspect of the experiment, we can see that as expected the metamer mismatch volumes are the largest in the centre of the OCS. Further, we can also see that the prior art method significantly underestimates the sizes of the mismatch volumes in the centre of the OCS.

The Logvinenko et al. method [5] employs an ineffient sampling strategy and consequently requires a very large number of 5-transition optimal spectra - preferably above 10000 samples - in order to estimate the size of $\mathscr{M}_5$. On the other hand, the better performing version of our algorithm usually requires as few as 1000 samples to accurately estimate the volume of the larger mismatch volume $\mathscr{M}$.

In Figures 5 and 6, we can see the graphical comparison of the mismatch volumes produced by our algorithm and the Logvinenko et al. method. The mismatch volume approximation $\mathscr{M}_5$ is indeed clearly contained within the theoretical limits of the mismatch volume $\mathscr{M}$ produced by our algorithm.

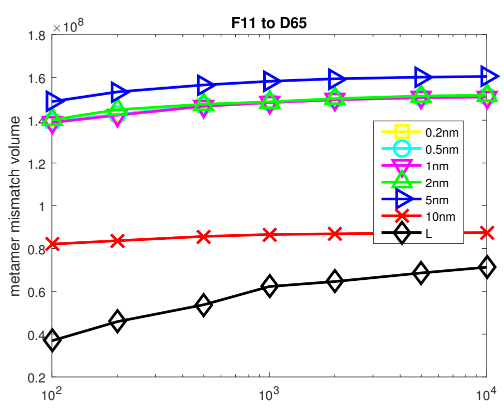

Figure 2. *As in Fig. 1, but for the change of illuminant from F11 to D65.*

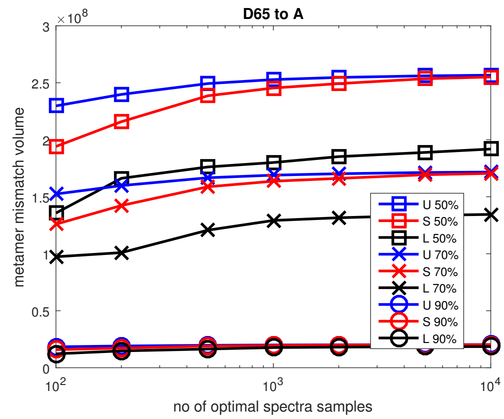

Figure 3. *Metamer mismatch volumes calculated for the flat grey reflectors with 50%, 70% and 90% reflectance for the the change of illuminant from D65 to A using the orthormal colour system spectra (U), original colour system spectra (S) with 1nm spectral sampling varying. (L) - method proposed by Logvinenko et al. [5].*

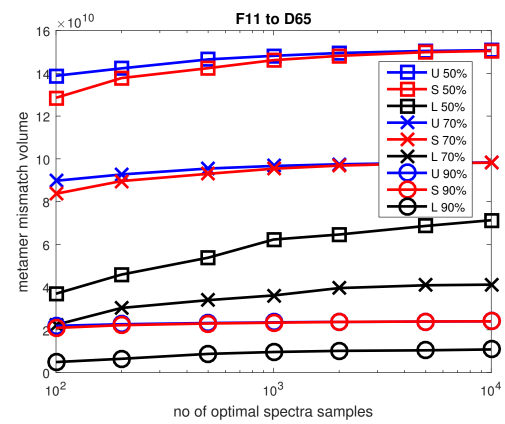

Figure 4. *As in Fig. 3, but for the change of illuminant from F11 to D65.*

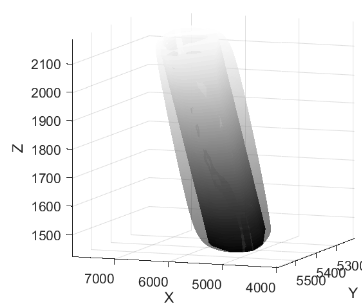

Figure 5. *Comparison of the metamer mismatch volumes calculated for the flat grey reflectance with 50% reflectance for the change of illuminant from D65 to A using 1nm spectral sampling with the corresponding volume calculated by the method proposed in[5]. Both methods use 10000 samples.*

Conclusions

Metamer mismatching is an important phenomenom in colour science. In this paper we proposed a novel algorithm for calculation of the theoretically maximum metamer mismatch volume. This, to our knowledge, is the first of its kind algorithm capable of calculating a precise maximum extent of these volumes. We have compared our work with the earlier prior art and conclude that the 5-transition approximation leads to a significantly smaller mismatch volume (sometimes above 50%!). Our algorithm is computationally efficient due to a simple linear programming formulation and a relatively small number of spherical samples required to provide precise estimates of the volumes of these objects.

References

[1] G. Wyszecki and W. Stiles, *Color Science: Concepts and Methods, Quantative Data and Formulae*. NY: John Wiley and Sons, 1982.

[2] G. Finlayson and P. Morovic, "Metamer Constrained Colour Correction," in *Proceedings of the 7th Color and Imaging Conference (CIC)*, 1999.

[3] ——, "Metamer Sets," *JOSA A*, vol. 22, no. 5, pp. 810–819, 2005.

[4] R. G. P. Urban, "Colour correction by calculating a metamer boundary descriptor," in *Proceedings of the 2nd Conference on Colour, Graphics, Imaging and Vision*, 2004.

[5] A. D. Logvinenko, B. Funt, and C. Godau, "Metamer mismatching," *IEEE Trans. Image Processing*, vol. 23, no. 1, pp. 34–43, 2014.

[6] E. Schrödinger, "Theorie der Pigmente von größter Leuchtkraft," *Annalen der Physik*, vol. 367, no. 15, pp. 603–622, 1920.

[7] G. West and M. Brill, "Conditions under which Shrödinger object colours are optimal," *Journal of the Optical Society of America*, pp. 1223–25, 1983.

[8] A. Logvinenko, "An object-color space," *Journal of Vision*, vol. 9, no. 11, pp. 1–23, 2009.

[9] N. Ohta and G. Wyszecki, "Theoretical chromaticity mismatch limits of metamers viewed under different illuminants," *J. Opt. Soc. Am.*, vol. 65, pp. 327–333, 1975.

[10] G. Finlayson, M. Mackiewicz, and H. J. Rivertz, "Computing the object colour solid using spherical sampling." in *Progress in Colour Studies*, London, UK, Sept. 2016.

[11] G. Marsaglia, "Choosing a point from the surface of a sphere," *Annals of Mathematical Statistics*, vol. 43, no. 2, pp. 645–646, 1972.

[12] C. Godau and B. Funt, "XYZ to ADL: Calculating Logvinenko's Object Color Coordinates," in *Proceedings of the 18th Color and Imaging Conference (CIC)*, 2010.

[13] G. Finlayson, M. Mackiewicz, and A. Hurlbert, "Making calculation of logvinenko's coordinates easy," in *Proceedings of the 20th Color and Imaging Conference (CIC)*, Los Angeles, US, Nov. 2012.

[14] G. Golub and C. van Loan, *Matrix Computations*, 4th ed. Johns Hopkins University Press, 2012.

[15] R. Hunt and M. R. Pointer, *Measuring Colour*, 4th ed. Wiley, 2011.

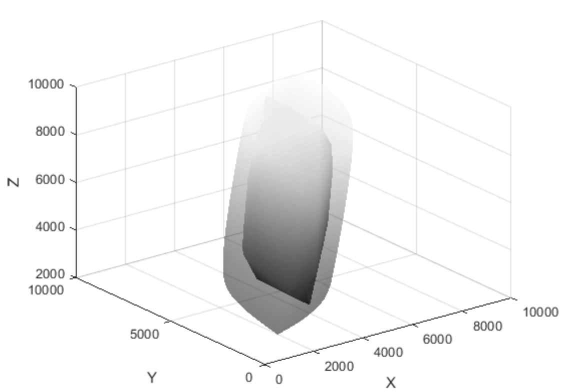

Figure 6. *As in Fig. 5, but for the change of illuminant from F11 to D65.*