

Distributions of Paramers and Paramer Mismatch Gamuts

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Abstract

Many coloration applications require the match of a reference color under specific viewing and illuminating conditions (primary conditions). Such so called metameric matches are, however, unlikely in practice due to various noise sources in most production processes. As a result the match is not perfect and the real colors are rather grouped around the reference color. In this paper we investigate such so called parameric colors under the assumption that they follow a normal distribution with the reference color as expectation value. We analyze corresponding distributions of parameric reflectances and calculate the distribution of colors if the viewing and illuminating conditions change (secondary conditions). A paramer mismatch gamut can be defined by the 95% boundary ellipsoid of this distribution. Based on this definition a formula is proposed to calculate its volume in terms of CIELAB unit. In the experiments we compare the volumes of 95% boundary ellipsoids for primary and secondary conditions and analyze the properties of a mismatch quantity that is defined as the ratio of these volumes.

Introduction

Reflectances are called metamers if they match under one viewing and illuminating condition but mismatch under other conditions. Sets of metameric colors were analyzed in various studies with respect to spectral properties and behavior under different conditions (Metamer Mismatch Gamut). [1, 2, 3, 4, 5, 6] This work was used to construct imaging and color reproduction systems [7, 8, 9, 10, 11].

In practice, however, it is very unlikely that two different reflectances behave metameric and achieve an exact tristimulus-match under the given viewing and illuminating conditions. Usually we have the situation that the tristimuli are very close to each other and the colors might be either indistinguishable or only slightly different. These reflectances are called *paramers*. The viewing and illuminating conditions for that the reflectances are paramers are denoted in this paper as *primary* or *parameric conditions*.

Of special interest are paramers that fall below the *just noticeable distance* (JND) to a reference color under parameric conditions and their behavior under different conditions. In practice the JND can also be replaced by a *tolerable distance* (TD). Reference colors are defined in many color-related standards and industrial applications. Especially for the coatings or printing industry such standards are highly important to ensure a consistent product quality. Usually additional color tolerances are given since it is unlikely that a production process is so precise that the resulting color matches the reference exactly.

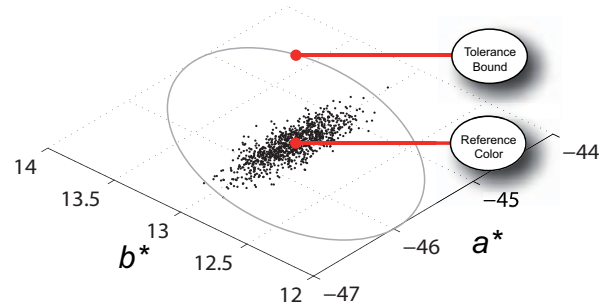


Figure 1. Colors resulting from a production process following a normal distribution with the reference color as mean value. The corresponding reflectances are called paramers.

In practice it is, however, unlikely that paramers resulting from a production process uniformly fill the tolerance ellipsoid for parameric conditions. Due to various noise sources in the production process the resulting colors are distributed rather normally with the reference color as mean (see Figure 1). In this paper we investigate paramers following a normal distribution for specific viewing and illuminating conditions (parameric conditions) and their mismatch gamut if viewed under different conditions (secondary or non-parameric conditions). We do not consider systematic color shifts from the reference color that may result from a biased production process for instance.

Our idea is to transform the normal distribution around the reference color from a color space for parameric condition to spectral space in order to calculate the corresponding paramers. A further transformation of this distribution into a color space for non-parameric conditions allows the calculation of a paramer mismatch gamut by boundary ellipsoids. All color- or spectral spaces for which a distribution is calculated are shown in Figure 2.

Calculating Distributions of Paramers

In this paper we use the vector representation of spectra by sampling them in the visible wavelength range at N equidistant wavelength. The parameric conditions are described by the spectral power distribution of an illuminant L_1 and the CIE color matching functions $\bar{x}_1, \bar{y}_1, \bar{z}_1$ of the 2 or 10 degree observer. The reference color $a \in \text{CIELAB}$ is given in the CIELAB color space. However, the concept explained below can be adapted to other color spaces if desired. The colors that correspond to paramers can be described by a random variable X_{CIELAB}^1 that follows a normal distribution $p(x)$

$$X_{\text{CIELAB}}^1 \sim \mathcal{N}(a, K_a) = p(x) \quad (1)$$

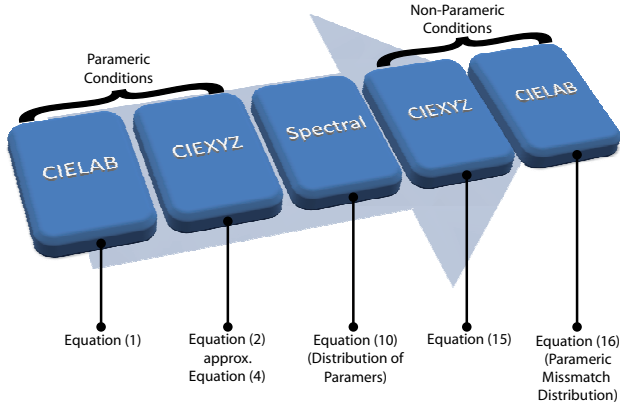


Figure 2. Transformation of distribution in this paper: The distribution of colors resulting from a production process is transformed through different color and spectral spaces from parameric conditions to non-parameric conditions.

where K_a is a 3×3 dimensional covariance matrix.

To calculate the paramers we need to transform this distribution in a first stage to the CIEXYZ color space. We denote the color space transformation from CIEXYZ space to CIELAB by \mathcal{L} and the inverse transformation by \mathcal{L}^{-1} . The resulting random variable in CIEXYZ space follows the distribution

$$X_{\text{CIEXYZ}}^1 = \mathcal{L}^{-1}(X_{\text{CIELAB}}^1) \sim \frac{p(x)}{|\det(J_{\mathcal{L}^{-1}}(x))|} \Big|_{x=\mathcal{L}^{-1}(y)} = p(y) \quad (2)$$

where $J_{\mathcal{L}^{-1}}(x)$ denotes the Jacobi matrix

$$J_{\mathcal{L}^{-1}}(x) = \begin{pmatrix} \frac{\partial \mathcal{L}^{-1}(x)}{\partial x_1} & \frac{\partial \mathcal{L}^{-1}(x)}{\partial x_2} & \frac{\partial \mathcal{L}^{-1}(x)}{\partial x_3} \\ \frac{\partial \mathcal{L}_1^{-1}(x)}{\partial x_1} & \frac{\partial \mathcal{L}_2^{-1}(x)}{\partial x_2} & \frac{\partial \mathcal{L}_3^{-1}(x)}{\partial x_3} \\ \frac{\partial \mathcal{L}_3^{-1}(x)}{\partial x_1} & \frac{\partial \mathcal{L}_3^{-1}(x)}{\partial x_2} & \frac{\partial \mathcal{L}_3^{-1}(x)}{\partial x_3} \end{pmatrix} \quad (3)$$

Since \mathcal{L}^{-1} is not linear the Jacobi matrix is not constant and the distribution $p(y)$ is not normal. To keep things simple we use a common technique in noise propagation and approximate the non-linear transformation \mathcal{L}^{-1} by the first two terms of a multidimensional Taylor series around the reference color $a \in \text{CIELAB}$. By substituting \mathcal{L}^{-1} in equation (2) by this truncated Taylor series we obtain a rough approximation of $p(y)$, which is again normal

$$p(y) \approx \mathcal{N}(\mathcal{L}^{-1}(a), J_{\mathcal{L}^{-1}}(a)K_a J_{\mathcal{L}^{-1}}(a)^T) \quad (4)$$

We want to calculate the distribution of paramers, i.e. reflectances in spectral space, so that the corresponding tristimuli for parameric conditions follow the normal distribution shown in equation (4). For this reason we need additional knowledge of the distribution of natural reflectances $p(r)$ since we have to estimate the distribution of N -dimensional paramers from a three dimensional distribution of tristimuli $p(y)$. For $N \gg 3$ there are a lot of possible solutions. To select the most reasonable out of these solutions we require additional knowledge. Inspired by the assumptions used by the Wiener spectral estimation we assume a normal distribution of natural reflectances $p(r) = \mathcal{N}(0, K_r)$, where K_r is a $N \times N$ dimensional covariance matrix of reflectances that can be estimated by a toeplitz matrix [12] or using a set of representative reflectances. It has to be noted that this assumption is

only a rough approximation of the real world. Investigations of the distribution of reflectances show rather a β -distribution or a mixture of normal distributions than a single normal distribution [13]. Nevertheless, the Wiener estimation is successfully used for spectral reconstruction in practice [14, 15] since it is a good compromise between simplicity and accuracy.

To increase our accuracy we can fit the normal distribution $p(r)$ also to reflectances resulting from colorant combinations used by the productions process.

To calculate the distribution of paramers we first calculate the distribution of metamers for the reference color a under parameric conditions. This can be done using Bayesian inference, since we know the prior distribution of reflectances $p(r)$ and the likelihood function

$$z = DLr = \Omega r \quad (5)$$

where $z \in \text{CIEXYZ}$ is the tristimulus, D is a $3 \times N$ dimensional matrix containing the color matching function $\bar{x}_1, \bar{y}_1, \bar{z}_1$ as row vectors and L is the $N \times N$ dimensional diagonal matrix containing the spectral power distribution L_1 as diagonal elements. The matrix $\Omega = DL$ is called the observer's lighting matrix.

This likelihood function can be used to derive the second prior distribution $p(z|r) \sim \mathcal{N}(\Omega r, K_\epsilon = 0)$. The posterior distribution $p(r|z)$ can be calculated by Bayes' theorem and is again normal [16]

$$X_r^{\text{meta}} \sim \mathcal{N}(Wz, K_r - W\Omega K_r) = p(r|z) \quad (6)$$

$$W = K_r \Omega^T (\Omega K_r \Omega^T)^{-1} \quad (7)$$

where X_r^{meta} describes the random variable that follows $p(r|z)$. If we transform this distribution to the CIEXYZ color space for the parameric conditions we obtain

$$\Omega X_r^{\text{meta}} \sim \mathcal{N}(\Omega Wz, \Omega(K_r - W\Omega K_r)\Omega^T) \quad (8)$$

$$= \mathcal{N}(z, 0) \quad (9)$$

If we draw values from this distribution they result all in z since the covariance matrix is zero. Hence $\mathcal{N}(0, K_r - W\Omega K_r)$ can be seen as the distribution of metameric black reflectances, i.e. all reflectances that result in a zero tristimulus, and $p(r|z)$ is the distribution of metameric blacks shifted by the fundamental metamer Wz . Based on the assumption of a normal distribution of reflectances the distribution $p(r|\mathcal{L}^{-1}(a))$ might be seen as a reasonable candidate of the distribution of metamers for the reference color a .

But how can this help us to derive a reasonable distribution of paramers? The distribution $p(r|\mathcal{L}^{-1}(a))$ needs to be updated by the distribution of all metamers for colors that follow the normal distribution shown in equation (4). For each of these colors z we need to calculate the fundamental metamer Wz and consider it as a paramer. Speaking in terms of distributions this means that we have to transform the normal distribution shown in equation (4) of tristimuli to spectral space by the linear function W and add this to the distribution of metamers for the reference color a . As a result and assuming that the production process noise that causes the deviation from the reference color a is statistically independent from metameric reflectances we have the following distribution

of paramers

$$X_r^{\text{para}} \sim \mathcal{N}(W\mathcal{L}^{-1}(a), K_r - W\Omega K_r + WK_p W^T) \quad (10)$$

$$W = K_r \Omega^T (\Omega K_r \Omega^T)^{-1} \quad (11)$$

$$K_p = J_{\mathcal{L}^{-1}}(a) K_a J_{\mathcal{L}^{-1}}(a)^T \quad (12)$$

where X_r^{para} is the random variable of paramers. Figure 3 shows the difference of metamers and paramers.

Transforming X_r^{para} to CIEXYZ color space for parameric conditions results in the normal distribution shown in equation (4), i.e.

$$\Omega X_r^{\text{para}} \sim \mathcal{N}\left(\Omega W \mathcal{L}^{-1}(a), \Omega(K_r - W\Omega K_r + WK_p W^T)\Omega^T\right) \quad (13)$$

$$= \mathcal{N}\left(\mathcal{L}^{-1}(a), J_{\mathcal{L}^{-1}}(a) K_a J_{\mathcal{L}^{-1}}(a)^T\right) \quad (14)$$

Paramer Mismatch Gamuts

Since paramers do not match even for parameric conditions the term *mismatch gamut* is somehow misleading. Nevertheless, we use the term *paramer mismatch gamut* to denote the gamut of paramers in a color space for non-parameric conditions according to the term *metamer mismatch gamut* for metamers.

The non-parameric conditions are described in this paper by the illuminant L_2 and an observer with the CIE color matching functions $\bar{x}_2, \bar{y}_2, \bar{z}_2$ of the 2 or 10 degree observer. To calculate such gamuts within the CIELAB color space we need to transform the distribution of paramers in a first stage into the CIEXYZ color space for non-parameric conditions. According equation (5) this is a simple linear transformation with a different lighting matrix $\tilde{\Omega} = \tilde{D}\tilde{L}$, where \tilde{D} is a $3 \times N$ dimensional matrix containing the color matching function $\bar{x}_2, \bar{y}_2, \bar{z}_2$ as row vectors and \tilde{L} is the $N \times N$ dimensional diagonal matrix containing the spectral power distribution L_2 as diagonal elements.

The resulting distribution in CIEXYZ color space for non-parameric conditions has the following form

$$X_{\text{CIEXYZ}}^2 \sim \mathcal{N}\left(\tilde{\Omega} W \mathcal{L}^{-1}(a), \tilde{\Omega}(K_r - W\Omega K_r + WK_p W^T)\tilde{\Omega}^T\right) \quad (15)$$

where X_{CIEXYZ}^2 is the random variable following this distribution.

In order to transform this distribution into the CIELAB color space we use the first two terms of the Taylor series for \mathcal{L} around the mean $\tilde{a} = \tilde{\Omega} W \mathcal{L}^{-1}(a)$ similar to the approximation in the previous section. The resulting distribution can be written as follows

$$X_{\text{CIELAB}}^2 \sim \mathcal{N}(\mathcal{L}(\tilde{a}), K_m) \quad (16)$$

$$K_m = J_{\mathcal{L}}(\tilde{a}) \left(\tilde{\Omega}(K_r - W\Omega K_r + WK_p W^T)\tilde{\Omega}^T \right) J_{\mathcal{L}}(\tilde{a})^T \quad (17)$$

where $J_{\mathcal{L}}(\tilde{a})$ is the Jacobi matrix of \mathcal{L} evaluated at position \tilde{a} . This normal distribution allows us to calculate the parameric mismatch gamut, which we define as the ellipsoid that contains 95% of the colors drawn from this distribution. Such ellipsoid is defined as follows:

$$(\mathcal{L}(\tilde{a}) - x) K_m (\mathcal{L}(\tilde{a}) - x) = \chi_3^2 \quad (18)$$

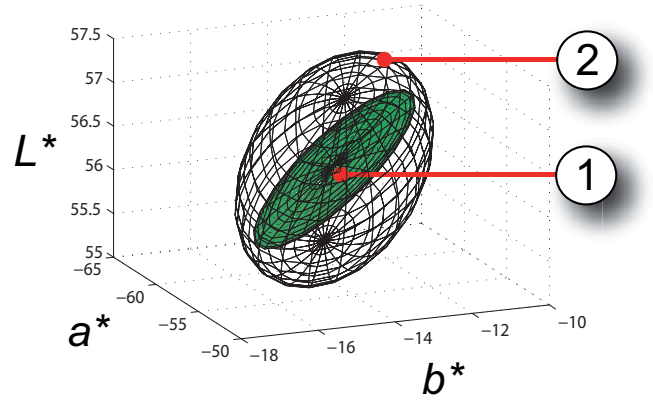


Figure 4. 95% boundary ellipsoids shown for non-parameric conditions corresponding to 1. metamer mismatch gamut 2. paramer mismatch gamut

where $\chi_3^2 = 7.81$ is the 95% quantile of the χ^2 distribution with three degrees of freedom.

It is also easy to calculate the volume of such ellipsoid in terms of CIELAB units using the following formula

$$\text{Vol}(K_m) = \frac{4\pi}{3} [\chi_3^2]^{3/2} \sqrt{\det[K_m]} \quad (19)$$

This allows us to relate the volume of the ellipsoid for parameric and non-parameric conditions in order to get a single quantity of mismatch

$$\text{Mismatch}(K_a, K_m) = \frac{\text{Vol}(K_a)}{\text{Vol}(K_m)} \quad (20)$$

Computational Experiments

We are interested in the increase of paramer mismatch gamuts if we increase the variance of the distribution for parameric conditions. For this reason we calculate the mismatch quantity defined in equation (20) for different parameters: We use the illuminants CIEA, CIED65 and CIEF11 for parameric as well as for non-parameric conditions. We chose the CIE 2 degree observer for all experiments. This results in nine different combinations of conditions.

To calculate the reference colors for parameric conditions we use the spectra of 1269 Munsell color chips freely available at the Information Technology Dept., Lappeenranta University of Technology, Finland. These spectra are also used for deriving the covariance matrix K_r of reflectances.

The covariance matrix K_a that describes the uncertainty in the CIELAB color space around the reference color is chosen as follows:

$$K_a = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (21)$$

where σ^2 varies between $\frac{0.1^2}{\chi_3^2}, \frac{0.2^2}{\chi_3^2}, \dots, \frac{3^2}{\chi_3^2}$ and $\chi_3^2 = 7.81$. This corresponds to 95% boundary spheres with an radius of $\Delta E_{ab}^* = 0.1, 0.2, \dots, 3$.

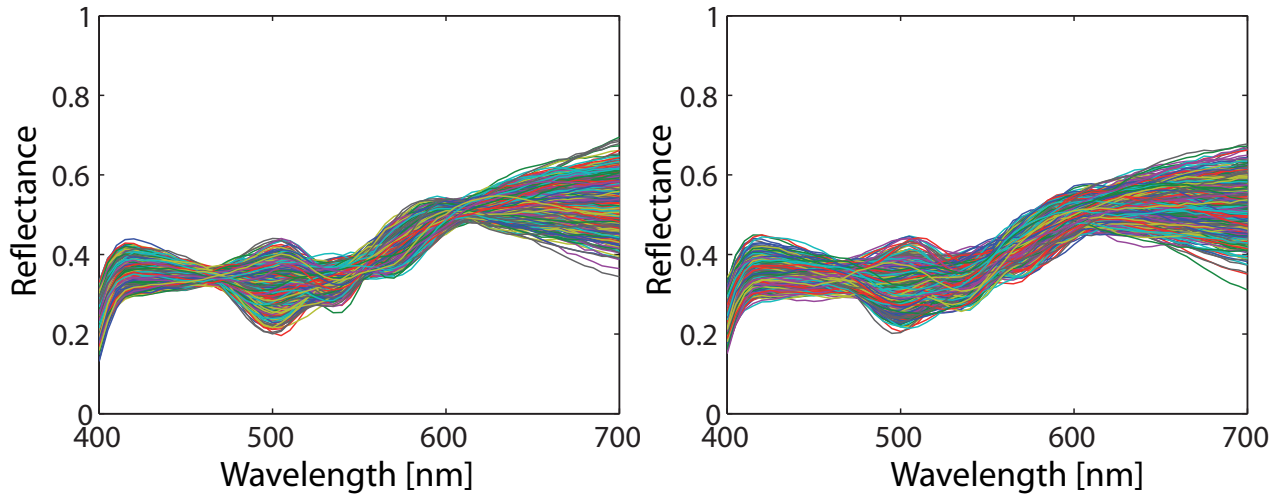


Figure 3. Left: Metamers drawn from the distribution shown in equation (6) for a specific reference color. Right: Paramers drawn from the distribution shown in equation (10) for the same reference color. The distribution in color space for parameric conditions is set in a way that 95% of the colors fall into a sphere with radius $\Delta E_{ab}^* = 1$ centered at the same reference color.

Results and Discussion

Figure 5 shows the results for the volume ratio defined in equation (20). The volume ratio is a number between 0 and 1. Smaller values show a larger mismatch. The number 0 is the special case if the paramers are in effect metamers, i.e. the covariance matrix K_a is zero and all colors match under parameric conditions. The number 1 occurs if the conditions do not change. Only six of the nine investigated combinations of conditions are shown in Figure 5 and 6. We also investigated the remaining combinations of illuminants CIEA - CIEA, CIED65 - CIED65 and CIEF11 - CIEF11. These result always in a volume ratio of 1 as expected, i.e. the volume of the 95% ellipsoid does not change if calculated by the covariance matrix K_a or by the covariance matrix K_m as shown in equation (17).

As can be seen from Figure 5 the volume ratio is nearly linearly related to the radius of the 95% boundary spheres under parameric conditions.

Figure 6 shows the relationship between the average 95% boundary ellipsoids' volumes for parameric and non-parameric conditions. Here the curves can be well parametrized by three parameters α, β, γ representing gain, offset and gamma (GOG), i.e.

$$\text{Vol}(K_m) = \alpha \text{Vol}(K_a)^\gamma + \beta. \quad (22)$$

The offset β is the average volume of the metamer mismatch gamuts and marks the ordinate intercept of the curves shown in Figure 5.

Figure 7 shows some examples of paramer and metamer mismatch gamuts. We can see that the size and shape of the mismatch ellipsoids depends on their location in color space. Also the gain in size of the paramer mismatch gamuts compared to the metamer mismatch gamuts varies at different locations in color space. This indicates that missing the reference color under parameric conditions caused by production process noise is not so critical for some reference color coordinates if the object is observed under non-parameric conditions. In this context *critical* means that the

possible color shifts for non-parameric conditions caused by missing the reference color for parameric conditions do not exceed the intrinsic uncertainty of metamerism.

It needs to be mentioned that the proposed method of calculating the distribution of paramers and paramer mismatch gamuts is only a rough approximation. In addition to simplified assumptions of the distribution of reflectances and the use of the first two terms of the Taylor series to approximate the non-linear color space transformations the physical properties of reflectances are not taken into account. These are mainly positivity and boundedness. As a consequence the paramer mismatch gamuts might be smaller than calculated in the experiments.

All these restrictions can be included into the calculations but the complexity would gain drastically. A possible solution would be the use of Monte Carlo methods similar to calculations of metamer mismatch gamuts [17]. The aim of this work was, however, to give a close formula of the distribution of paramers and the volume of the paramer mismatch gamut. It should allow the reader to reimplement the formula with the simplest means to get an idea about the paramers and the paramer mismatch gamut. The formula can be seen as a compromise between accuracy and complexity.

Conclusion

In this paper paramers and paramer mismatch gamuts are investigated assuming that the parameric colors for specific viewing and illuminating conditions (parameric conditions) follow a normal distribution with a reference color as expectation value. To calculate the distribution of corresponding parameric reflectances (paramers) this distribution is transformed into spectral space. A further transformation into a color space for different viewing and illuminating conditions allows the determination of the distribution of corresponding colors for non-parameric conditions. Paramer mismatch gamuts are calculated using 95% boundary ellipsoids of such distributions. A simple formula allows the

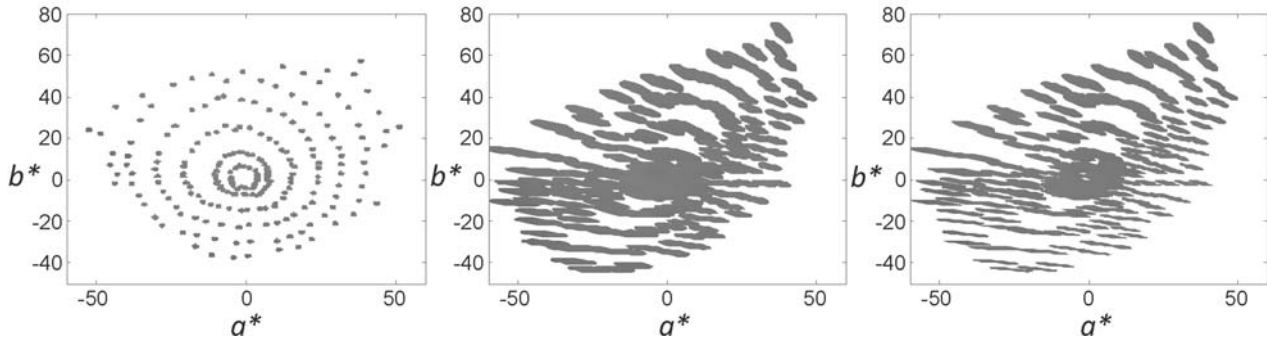


Figure 7. Left: 95% boundary spheres (radius: $\Delta E_{ab}^* = 1$) of Munsell colors in the lightness range $55 < L^* < 65$ for illuminant CIED65 under parameric conditions. Middle: Corresponding paramer mismatch 95% boundary ellipsoids and right: metamer mismatch 95% boundary ellipsoids for illuminant CIEA.

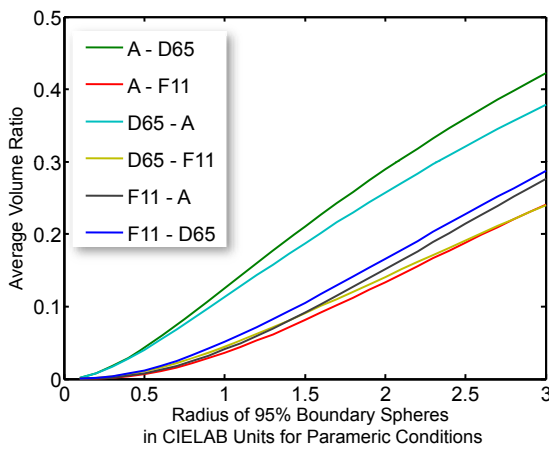


Figure 5. Parameric mismatch in terms of the volume ratio between 95% boundary ellipsoids for parameric and non-parameric conditions. The smaller the value the larger the mismatch.

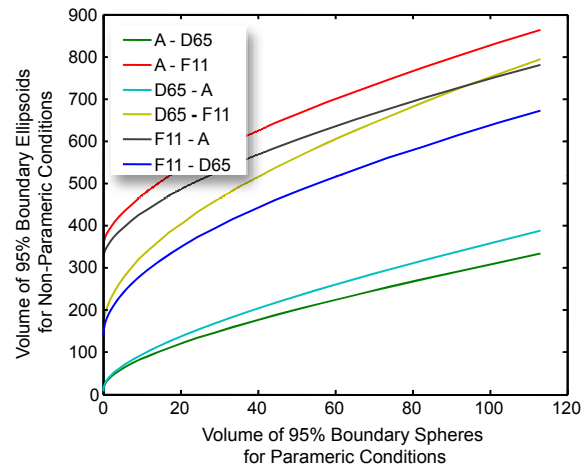


Figure 6. Volume of 95% boundary spheres for parameric conditions plotted against the average volume of 95% boundary ellipsoids for non-parameric conditions. The values are shown in cubic CIELAB units.

calculation of the volume of these gamuts in terms of CIELAB units. Experiments show a functional relationship between the average volumes of the 95% boundary ellipsoids for parameric and non-parameric conditions that can be described by a gain, offset, gamma (GOG) model. The offset represents the average volume of all metamer mismatch gamuts corresponding to the reference colors. The volume of the paramer mismatch gamuts and their gain in size compared to corresponding metamer mismatch gamuts depends on the location of the reference color in color space.

Acknowledgements

The author thanks the Deutsche Forschungsgemeinschaft (German Research Foundation) for the sponsorship of this project.

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Author Biography

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