Mathematical Discontinuities in CIEDE2000 Color Difference Computations

*Gaurav Sharma**, *Wencheng Wu*+, *Edul N. Dalal*+, *Mehmet U. Celik**

**ECE Dept. Univ. of Rochester Rochester, NY 14627-0126*

+*Wilson Center for Research & Tech. Xerox Corp., Webster, NY 14580*

Abstract

We examine mathematical properties of the CIEDE2000 color difference formula and illustrate that the CIEDE2000 color difference is not a continuous function of the CIELAB color pairs between which the difference is calculated. Particularly, we illustrate that ambiguities in the computation of mean hue angle and hue angle differences of color pairs contribute to a discontinuity in the formula for color pairs chosen 180-degrees apart in CIELAB hue. We analyze and characterize the discontinuity and give a visual presentation that allows better appreciation of its nature. For color pairs under 5 CIELAB delta E units apart, the maximum discontinuity magnitude is bounded by 0.274. For color pairs further apart in CIELAB, the magnitude of the discontinuity rises sharply. The results indicate that care should be exercised in using the formula in Taylor series approximations and that its use should be restricted to small color differences as recommended by the CIEDE2000 standard.

1. Introduction

Color difference equations are frequently used to quantify color differences among color samples in perceptually relevant magnitudes. The CIELAB and CIELUV color spaces [1] were standardized by the CIE in 1976 as approximately uniform color spaces in which Euclidean distances are close to perceptually uniform. Since the perceptual uniformity of these spaces is less than desired, alternative methods of color difference computation have been proposed and standardized. The most notable among these are the CMC [2], CIE94 [3] and the relatively recent CIEDE2000 [4] color difference formulae. Each of these formulae attempts to improve the perceptual uniformity of the computed color differences by decomposing the Euclidean distance in CIELAB into components corresponding to differences in lightness, hue, and chroma; which are then recombined in a weighted mean squared fashion, where the weights are dependent on the location of the samples in color space. These location dependent weights are referred to as *weighting functions*. The CIEDE2000 color difference formula introduces an additional "rotation term" which is active in the blue region and attempts to capture the observed elongation of equi-perceptual-magnitude ellipses in this region in the direction of observed hue-nonlinearity in CIELAB [5].

The primary focus of the research on improved color difference formulae is the improvement of perceptual uniformity. However, the mathematical properties of these formulae are also of significant interest, particularly, their continuity and differentiability. Specific examples of analysis/applications which rely on these mathematical properties include gradient-based optimization methods [6], methods for the analysis of error propagation based on Taylor-series approximations [7, 8], and perceptually meaningful metrics for color accuracy derived from small error approximations using Taylor series approximations [9]. Symmetry of the computed color difference with respect to the (sometimes arbitrary) designation of members of a color pair as a reference and a sample is also desirable. The symmetry is not only esthetically desirable, but also affords practical convenience because the choice of reference and sample designation need not be specified in scenarios where it is not central to an experiment.

In the early development and modification of the CIELAB and CIELUV spaces, the functional transformations chosen for the conversion from CIEXYZ to CIELAB/CIELUV were explicitly designed to ensure both continuity and continuity of the first derivative [10]. In addition, symmetry was inherently ensured by the computation of the difference as a Euclidean distance between the points corresponding to the individual colors in a pair. Subsequently developed color difference formulae based on CIELAB do not use an Euclidean distance computation. These therefore are not assured of inheriting the symmetry and continuity properties of the CIE 1976 ΔE^*_{ab}. In particular, the CMC color difference formula [2] uses *weighting functions* that are dependent on the color coordinates of the chosen "standard" (or reference) color. The CMC computation is therefore asymmetric. The same is true for the basic CIE94 [3] color difference computation, though a method of symmetrization was also recommended in the standard. The symmetrization uses the mean chroma for determination of *weighting functions*, which makes it invariant to the choice of the standard and sample, thereby ensuring symmetry. Both the CMC color difference formula [2] and the CIE94 [3] color difference formula, preserve continuity. The CIEDE2000 formula extensively uses the same symmetrization technique as the one proposed for the CIE94 color difference, all *weighting functions* are defined in terms of means for the reference and sample color coordinates, which ensures symmetry. However, as illustrated in subsequent sections, the CIEDE2000 formula provides a color difference that is not a continuous function of the chosen pair of colors. Ironically, the major part of the discontinuity arises from the computation of mean-hue, a step that was probably introduced to symmetrize the *weighting functions* (and consequently the formula).

2. Computation of the CIEDE2000 Color-Difference

A description of the CIEDE2000 color difference formula can be found in [4, 5] and a complete algorithmic statement of the computation, along with implementation notes and test data can be found in [11]. For brevity, we will only highlight particular equations necessary for our discussion of the mathematical properties. The description here is *conceptual* and not intended to be a guide for implementation of the formula.

The CIEDE2000 color difference formula is based on the CIELAB color space. Given a pair of color values in CIELAB space L_1^*, a_1^*, b_1^* and L_2^*, a_2^*, b_2^*, we denote the CIEDE2000 color difference between them as

$$\Delta E_{00}(L_1^*, a_1^*, b_1^*; L_2^*, a_2^*, b_2^*) = \Delta E_{00}^{12} = \Delta E_{00} \qquad (1)$$

The formula for the computation of the difference can be expressed in the form shown in Eqn (2) (*see next page*), where k_L, k_C, and k_H are application-dependent parametric parameters (assumed to be 1 in our subsequent discussions); S_L, S_C, and S_H are lightness, chroma, and hue dependent *weighting functions*, respectively; and R_T is an additional *weighting function* that depends on chroma and hue. The terms $\Delta L'$, $\Delta C'$, $\Delta H'$ are the lightness, chroma, and (metric) hue difference, respectively (computed in a modified version of the CIELAB space).

To ensure symmetry, i.e., to assure $\Delta E_{00}^{12} = \Delta E_{00}^{21}$, the lightness, chroma, and hue dependent *weighting functions* are computed in terms of the average Lightness, Chroma, and hue (angle) of the two colors in the (modified) CIELAB space. We will consider specifically the terms $\Delta H'$ and the *weighting function* S_H which are computed as

$$\Delta H' = 2\sqrt{C_1' C_2'} \sin\left(\frac{\Delta h'}{2}\right) \qquad (3)$$

$$T = 1 - 0.17\cos(\bar{h}' - 30^\circ) + 0.24\cos(2\bar{h}') + 0.32\cos(3\bar{h}' + 6^\circ) - 0.20\cos(4\bar{h}' - 63^\circ) \qquad (4)$$

$$S_H = 1 + 0.015\bar{C}'T \qquad (5)$$

where C_1', C_2' are the chroma values for the samples, $\Delta h'$ is the hue angle difference between the samples, $\bar{h}'$ is the mean hue angle, and $\bar{C}'$ is the mean chroma.

The mean hue angle $\bar{h}'$ and the hue angle difference $\Delta h'$ are computed in a geometric sense, instead of a purely arithmetic computation. Thus $\Delta h'$ is the angle between the line segments drawn at modified hue angles h_1' and h_2' and the mean hue angle $\bar{h}'$ is the hue angle of the angular bisector for this angle. This is illustrated in Fig. 1. The two dots in this figure labeled 1, 2 represent sample colors projected onto the a'-b^* plane. Line segments have been drawn from the origin in the plane to each of the colors' projections. The modified hue angle of each color is the angle that the line segment makes with respect to the a' axis measured in the counter-clockwise direction as indicated by the arcs labeled h_1' and h_2'. The angle between the two line segments represents the hue angle difference $\Delta h'$ as labeled (measured from 1 to 2). The angular bisector of the two line segments is shown as the broken line segment with a pointed arrow. The mean hue $\bar{h}_{12}'$ corresponds to the angle that this bisector makes with respect to the a' axis measured in the counter-clockwise direction as indicated by the arcs labeled $\bar{h}_{12}'$. The geometric notion of the angle difference always results in an absolute value for the hue angle difference $\Delta h'$ under 180° and a mean hue angle between 0 and 360° which corresponds to the bisector of the smaller of the angles between the two line segments corresponding to h_1' and h_2' (this is relevant in situations, where the absolute value of the arithmetic hue angle difference exceeds 180°, i.e. $|h_2' - h_1'| > 180^\circ$).

3. Discontinuities in the CIEDE2000 Color Difference Formula

The process for computation of mean hue and the hue angle difference outlined in the previous section have an inherent ambiguity for hues that are exactly 180° apart in hue. This contributes directly to a discontinuity in the computation of mean hue and the hue angle difference. Using the geometric interpretation of the mean hue, the discontinuity introduced in its computation is readily seen in Fig. 2. In the figure, three sample colors labeled 1, 2, and 3 are plotted on the a'-b^* plane such that the modified hue

$$\Delta E_{00}^{12} = \sqrt{\left(\frac{\Delta L'}{k_L S_L}\right)^2 + \left(\frac{\Delta C'}{k_C S_C}\right)^2 + \left(\frac{\Delta H'}{k_H S_H}\right)^2 + R_T \left(\frac{\Delta C'}{k_C S_C}\right)\left(\frac{\Delta H'}{k_H S_H}\right)} \quad (2)$$

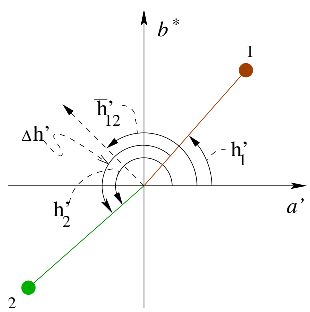

Figure 1: *Geometrical illustration of the computation of mean hue and hue angle difference.*

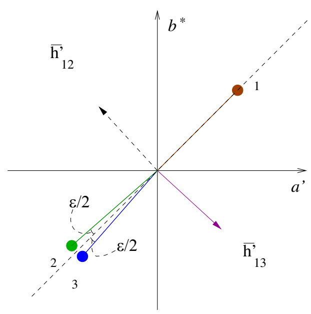

Figure 2: *Geometric illustration of the* $180°$ *discontinuity in mean hue computation.*

angles of 2 and 3 are close to $180°$ apart from the modified hue angle of 1 with the absolute hue angle difference between 1 and 2 just under $180°$ and the the absolute hue angle difference between 1 and 3 just over $180°$. The dashed line segment with the arrow represents the mean hue $\bar{h'_{12}}$ for samples 1 and 2 and the solid line segment with the arrow represents the mean hue $\bar{h'_{13}}$ for samples 1 and 3. From the figure, it is clear that the small perturbation from 2 to 3 produces a change in mean hue of over $180°$. An identical argument illustrates that the computation of hue angle difference suffers from a similar discontinuity.

The discontinuity resulting from the mean hue computation impacts the terms in the CIEDE2000 equations that use this mean hue in further computations. At the lowest level, this occurs in the term T and in Eqn. (4) presented above[1]. The discontinuity in the computation of the hue angle difference causes a sign reversal in $\Delta H'$ which in turn creates a discontinuity in Eqn. (2) in the rotation term involving R_T that uses the signed metric hue difference (as opposed to other terms which use a squared difference for which the hue angle difference discontinuity is eliminated). Since these terms are used in combination with other terms in the computation of the final color difference ΔE_{00}, the magnitude of the discontinuity in ΔE_{00} cannot be directly and completely understood based on the discontinuity in these terms. We do note [2] that the main contribution to the discontinuity in ΔE_{00} arises from the discontinuity in mean hue computation, through the term

$$\left(\frac{\Delta H'}{k_H S_H}\right)^2 \quad (6)$$

in Eqn. (2), which inherits the discontinuity from the term T in Eqn. (4). Based on the definition of $\Delta H'$ in Eqn. (3) it is clear that the magnitude of the discontinuity increases with increasing chroma values C'_1 and C'_2.

3.1. Discontinuity Loci

The ΔE_{00} color difference in Eqn. (1), is a non-negative, real-valued function defined over a 6-dimensional space, where three color coordinates each, of a pair of colors, constitute the 6 dimensions. When CIELAB coordinates are used for the pair of colors, the loci of points in the 6-dimensional space for which the ΔE_{00} color difference is discontinuous due to mean-hue and hue-difference computations can be inferred from Fig. 4 as the points located $180°$ apart in CIELAB hue, i.e., points satisfying

$$a_1 b_2 = -a_2 b_1 \quad (7)$$

These points define a 5-dimensional manifold in the 6-dimensional space.

The use of CIELCH coordinates for the color pairs yields a simpler representation for the discontinuity loci. In this case, the loci of points at which the ΔE_{00} color difference is discontinuous can be represented in a 2-D h_1 vs. h_2 plane where the two axes represent the CIELAB hues h_1 and h_2 of the two colors. Fig. 3 illustrates the discontinuity loci in the h_1 vs. h_2 plane. The two line segments $h_2 = h_1 + 180°$ and $h_2 = h_1 - 180°$ shown in the diagram represent the loci of points at which the CIEDE2000 color difference is discontinuous due to the discontinuity in the computation of mean hue or hue difference. In general, these loci correspond to discontinuities in the CIEDE2000 color difference irrespective of the values of

[1]The discontinuity also impacts other terms not described here. However, their contribution to overall discontinuity is negligible.

[2]See [11] for additional details.

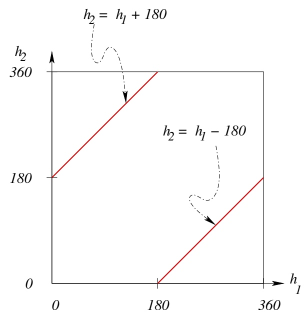

Figure 3: *Discontinuity Loci of CIEDE2000 color difference in h_1 vs. h_2 plane (hue values for the two colors).*

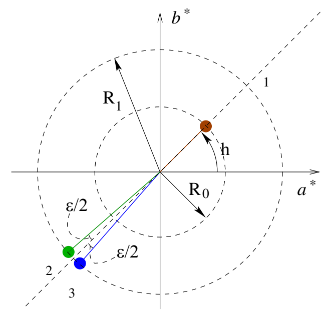

Figure 4: *Color configuration for empirical evaluation of ΔE_{00} discontinuity due to hue-difference computation.*

lightness and chroma for either color, provided the chroma is non-zero for both colors. The dependence on the lightness and chroma variables may therefore be suppressed for the purpose of this discussion. Each line segment in the diagram represents a 5-dimensional hyperplane in the 6-dimensional space over which the CIEDE2000 color difference formula is discontinuous. This represents a significant territory that over which the CIEDE2000 color difference is discontinuous.

3.2. Discontinuity Magnitude

The complicated nature of the ΔE_{00} equations coupled with the multiple discontinuities (at the same loci) makes analytical determination of the discontinuity magnitude somewhat tedious. Instead, we adopt an empirical approach. We begin by selecting a suitable set of colors in CIELAB space that illustrate the discontinuity. From Fig. 2, it is apparent that the discontinuities in the computation of the mean hue and the hue angle difference arise only for colors that are $180°$ apart in CIELAB hue angle. From the CIEDE2000 equations it is also clear that the magnitude of the discontinuity in ΔE_{00} is not a fixed value but will depend on the choice of the pair of color locations. For estimating the magnitude of discontinuity in the ΔE_{00} formula arising due to these two discontinuities, we use the specific configuration of colors shown in the a^*-b^* plane in Fig 4 [3]. The point labeled as 1 represents a reference color located at CIELAB hue h and having CIELAB chroma R_0. The points labeled 2 and 3 represent two samples having CIELAB chroma R_1 and located at hue angles $h + 180° - \epsilon/2$ and $h + 180° + \epsilon/2$, respectively. For our computations, we use $\epsilon/2 = 10^{-6}$radians. In general, $\epsilon/2$ should be the smallest possible value whose impact is not masked by the limited precision of computation.

The magnitude of the discontinuity in the CIEDE2000 color difference formula is then estimated as the absolute value of the difference between the CIEDE2000 color difference ΔE_{00}^{12} between 1 and 2 and the CIEDE2000 color difference ΔE_{00}^{13} between 1 and 3

$$d_{\Delta E}(h, R_0, R_1) = |\Delta E_{00}^{12} - \Delta E_{00}^{13}| \qquad (8)$$

4. Visualization of the CIEDE2000 Discontinuity Magnitude

The three-dimensional function $d_{\Delta E}(h, R_0, R_1)$ corresponding to the discontinuity magnitude can be visualized along two dimensions at a time by fixing the third dimension. Fig. 5 illustrates the GUI of a MATLAB program that allows exactly such a visualization. One of the three variables corresponding to hue h, reference chroma R_0 or sample chroma R_2 may be selected and set to a desired value. The variation of the discontinuity magnitude $d_{\Delta E}(h, R_0, R_1)$ as a function of the other two variables is then plotted in the form of a 3-D mesh plot. The specific plot shown in Fig. 5 shows the variation in discontinuity magnitude as a function of reference chroma and sample chroma (each ranging from 0.1 to 2.5) for a fixed value of the hue angle of $143°$. From the plot, one can see that for the chosen hue angle, the the discontinuity magnitude increases monotonically with increase in sample or reference chroma value.

Fig. 6 shows the variation in discontinuity magnitude as a function of hue angle (ranging from $0°$ to $360°$) and sample chroma (ranging from 0.1 to 2.5) for a fixed value of the reference chroma of 2.5. From the plot, it is apparent that for the chosen reference chroma, the discontinuity magnitude increases with increasing value of the sample chroma and has peaks corresponding to four distinct hue values of roughly $37°$, $87°$ and $143°$; with the highest peak at $143°$.

By manipulating the visualization GUI parameters, the variation of the CIEDE2000 discontinuity magnitude can

[3]Since the L^* values of the chosen pairs do not contribute to the discontinuity or influence its magnitude, we fi x the L^* values at 50 for our reference and samples (any other value may be equivalently used).

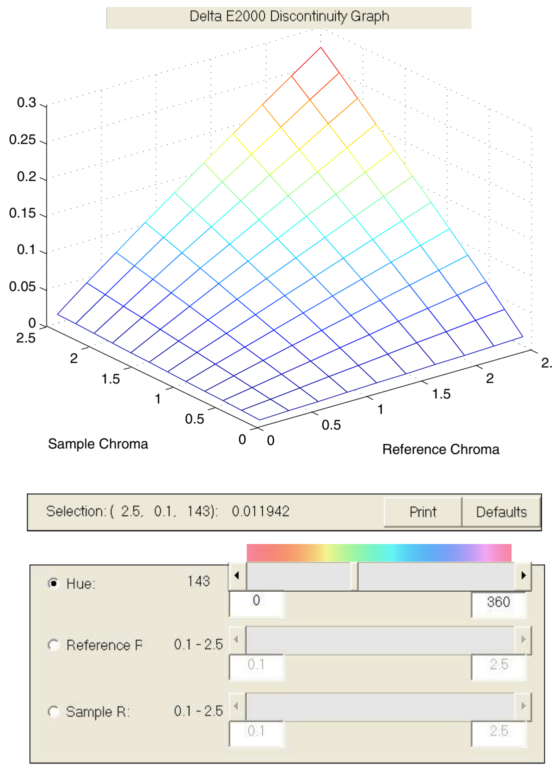

Figure 5: MATLAB GUI for visualizing the discontinuity-magnitude $d_{\Delta E}(h, R_0, R_1)$. The mesh-plot within the figure illustrates the discontinuity magnitude as a function of reference chroma R_0 and sample chroma R_1 for a hue angle $h = 143°$.

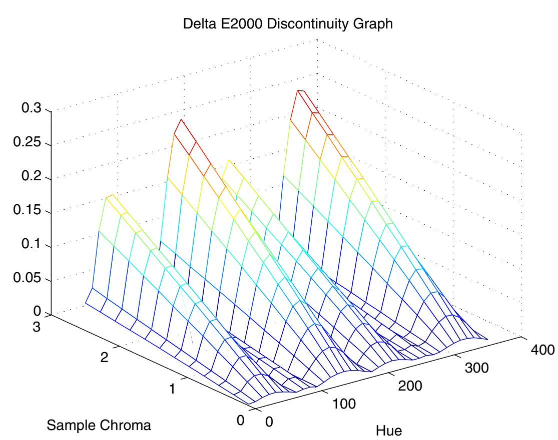

Figure 6: Mesh plot of discontinuity-magnitude $d_{\Delta E}(h, R_0, R_1)$ as a function of sample chroma R_1 and hue angle h for a reference chroma value $R_0 = 2.5$.

be studied extensively. The discontinuity magnitude $d_{\Delta E}(h, R_0, R_1)$ has the following characteristics:

- $d_{\Delta E}(h, R_0, R_1)$ increases monotonically with increasing R_0, for all values of h and R_1.
- $d_{\Delta E}(h, R_0, R_1)$ increases monotonically with increasing R_1, for all values of h and R_0.
- $d_{\Delta E}(h, R_0, R_1)$ is an oscillating function of h for all values of R_0 and R_1. The function has local maxima at four hue values of roughly $37°$, $87°$ and $143°$; with the highest value at $143°$.

By further analyzing the components contributing to the discontinuity in the CIEDE2000 computation [11], one can conclude that the discontinuity in mean hue computation is the dominant contributor to the overall discontinuity magnitude observed and the contribution of the discontinuity in hue difference computation to the overall discontinuity magnitude is negligible in comparison and localized in the blue hue region ($\bar{h}' = 270°$) corresponding to the rotation term involving R_T in Eqn. (2).
For general applications of the ΔE_{00} formula, it is also helpful if we can bound the **maximum discontinuity** magnitude that may be encountered. An empirical evaluation was therefore performed, where for each selected hue angle, we determined the points with the maximal discontinuity in ΔE_{00} that were within a CIELAB ΔE^*_{ab} of 5 units from each other (equivalently, within 5 CIELAB chroma units from each other). In each of the cases tested, the maximal discontinuity occurred for a configuration of colors with equal "reference" and "sample" chroma values of 2.5, and $180°$ apart in CIELAB hue angle. This configuration is included in Figs. 5 and 6 and corresponds to the point $R_0 = R_1 = 2.5$ and $h = 143°$. Globally, for colors that are within 5 CIELAB ΔE^*_{ab} units from each other, the discontinuity in CIEDE2000 color difference ΔE_{00} is under 0.2734.

5. Conclusions and Discussion

The CIEDE2000 color difference is a discontinuous function. The significant discontinuity in the computation of the CIEDE2000 color difference manifests itself only for samples that are $180°$ apart in hue, i.e., located in opposite quadrants of the $a^* - b^*$ plane. Since the CIEDE2000 formula is applicable primarily for small color differences, both samples will typically be close together. Therefore, the only situation under which they may lie in opposite quadrants is for the case of colors close to gray. These have a low value of chroma and therefore the magnitude of the discontinuity will be small in practical applications. As illustrated in the previous section, if the samples are under $5\Delta E^*_{ab}$ units apart, the discontinuity in CIEDE2000 color difference ΔE_{00} is under 0.2734, which is small in comparison to color differences encountered in a number of applications, but not negligible. If the samples are $1\Delta E^*_{ab}$ unit apart, the discontinuity magnitude is smaller than 0.0119, which is negligible in most practical situations.

The major source of the discontinuity in the CIEDE2000 color difference computation is the discontinuity in the the computation of mean hue, which is necessary for making the hue dependent weighting function S_H symmetric (as a prelude to a symmetric color difference formula). By sacrificing the symmetry and using the hue of the reference color for the determination of *weighting functions*, this source of discontinuity may be eliminated leaving only other sources of discontinuity that are significantly smaller. In the presence of asymmetric weighting functions, the overall color difference formula can still be mathematically symmetrized by other means, for instance by defining a new formula corresponding to $(\Delta E^{12}_{00} + \Delta E^{21}_{00})/2$.

The discontinuities in the CIEDE2000 color difference computation may not be a major concern in several industrial applications, where color configurations with discontinuities may not occur or experimental variation may be much larger than encountered discontinuity magnitudes. However, the discontinuities do preclude the use of the formula in analysis based on first order Taylor series approximations and in gradient based optimization methods that not only require continuity of the function but also continuity of the first derivative.

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