Group Theoretical Invariants in Color Image Processing

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Abstract

Many image formation processes are complex interactions of several sub-processes and the analysis of the resulting images requires often to separate the influence of these sub-processes. An example is the formation of a color image which depends on the illumination, the properties of the camera and the objects in the scene, the imaging geometry and many other factors. Color constancy algorithms try to separate the influence of the illumination and the remaining factors and are thus typical examples of the general strategy. An important tool used by these methods are invariants ie. features that do not depend on the state of one (or several) of the sub-processes involved. Illumination invariants are thus features that are independent of illumination changes and depend only on the remaining factors such as material and camera properties.

We introduce transformation groups as the descriptors of the sub-processes mentioned above. We then show how they can be used to calculate the number of independent invariants for a given class of transformations. We also show that the theory is constructive in the sense that there are symbolic mathematics packages that can find the invariants as solutions to systems of partial differential equations.

We illustrate the general theory with applications from color computer vision. We will describe the construction of invariants from the dichromatic and the Kubelka-Munk reflection models in detail. Space does not permit us to describe the detailed derivation of illumination invariants from PCA models of illumination spectra but it can be shown that the construction of the invariants follows the same mathematical procedure.

1. Introduction

Many pattern recognition models assume that the available measurements are the result of a number of different interacting processes. A standard model for color image formation assumes, for example, that the final color image depends on the characteristics of the illumination, the scene geometry, the properties of the materials in the scene and the characteristics of the camera. In typical applications we are only interested in one or a few of these factors while the others should be ignored. This is the basic idea of invariants in pattern recognition. In the color image processing example mentioned above we may want to ignore the influence of the illumination source or the geometric relations between the camera and the scene. Invariance based mechanisms are highly successful in biological systems and we apply them all the time without even being aware of them. They are obviously one important tool to stabilize our perception of the world and to simplify our interpretation of it. Typical examples from human perception are:

Color Constancy: We routinely ignore the effect of variations of the spectral characteristics of the illuminants

Orientation Invariance: We can compensate to a certain extend the influence of changes of viewing directions on the perceived image

Shape Changes: A typical, and very important example, of this type of invariance is face perception. We can recognize a persons face independent of large variations of its shape, often connected to emotions like joy, sadness etc.

Permutation invariance: Often we can recognize a collection of item independent of the order in which they are arranged.

Without conditional constraints it is very difficult to solve such invariance problems since without such constraints we would need something like a table that collects all the different cases that we would like to ignore. The solution to this problem is that invariance problems can be more easily solved if there is a law or a rule describing it. This law or rule can often be described in the framework of group theory and mathematical theories developed in the last 100 years provide the tools to investigate and solve these problems. Some of the first attempts to apply group theoretical tools to investigate properties of biological systems (and especially perceptional invariants) go back, at least, to Wiener. Some of the main ideas in this area are described in his book on Cybernetics [17, 10, 11] and have been reinvented many times since then.

In this paper we illustrate how one part of group theory, the Lie theory of differential equations, can be applied to answer the following two questions:

- How many invariants exist for a given problem?
- How can we find all of them?

We will describe in detail two applications from color image processing to illustrate how these techniques can be used:

- Geometry invariants from the dichromatic reflection model
- Geometry invariants from Kubelka-Munk models

Illumination invariants from conical illumination spaces can be derived within the same framework but we will not describe these results in detail. We will give a brief description of the underlying mathematical theory, develop the color models as far as necessary and illustrate the usage of symbolic mathematical software packages like Maple in obtaining all the invariants of a given problem.

Before we come to a detailed description of the approach we want to make some general remarks:

- The theory we will describe is very general and the color applications described here are only one illustration of its usage, among many others
- Our primary interest is not in practical applications but in the basic insight gained from the derivation. We will further comment on this at the end of the paper.
- A last issue concerns the question if the mathematical apparatus is really necessary. One answer is of course that much of the group theory can be avoided if the primary goal is to derive the color related invariants. On the other hand we feel that the current problem provides a good illustration of the power of the general theory. A final decision if the result was worth the additional effort depends on the preferences of the reader.

2. Construction of Invariants

The basic mathematical tool to investigate invariants is the concept of a transformation group. We recall that a group is a set of elements such that the each element has an inverse and each combination of two group elements gives another element in the same set. A transformation group is a special group in which all elements are transformations defined on a set. We will only consider cases where the group elements are matrices and the sets on which they operate are subsets of real Euclidean vector spaces. We use the following notation: The group of transformations is denoted by G and the set on which the transformations operate is X. The elements in G and X are $M, N, \ldots$ and $x, y, \ldots$ and the notation for the transformation group is (G, X). The transformation is written as $x \mapsto M\langle x\rangle$. The two simplest examples are the shifts operating on the real line and the rotations operating on the circle. For the rotations this gives

$$\left(\mathrm{SO}(2), \mathbb{R}^2\right); (M, x) \ \mapsto \ M\langle x\rangle \ = \ Mx$$

with a 2-dimensional vector x, rotation matrix M and Mx the matrix-vector multiplication. A subgroup is a subset of a group that is also a group. A one-parameter subgroup is a subgroup that depends on only one parameter and in the following this means that we can write its elements as matrix exponentials $M_t = \mathrm{e}^{tX}$. Generalizing we say a group is an $k-$parameter group if there are k matrices $X_1, \ldots, X_k$ such that all group elements have the form $M = \mathrm{e}^{t_1X_1+\ldots+t_kX_k}$. The matrices M form the Lie-group and the matrices $t_1X_1 + \ldots + t_kX_k$ the Lie-algebra. As a final concept we need the connection between the Lie-algebra and differential operators: take a function f defined on the set X. For a group element M we can define the new function $f_M(x) = f(M\langle x\rangle)$ and for a one-parameter group we can consider $f_t(x) = f(M_t\langle x\rangle)$ as a function of t. We can then compute the derivative $\partial f/\partial t\|_{t=0}$ and we see that every one-parameter group defines a differential operator.

With these preparations we can now describe the main mathematical results used in the following:

- *A function f is an invariant for a group G if for all elements $M \in G$ we have $f_M(x) = f(x)$.*
- *If the set X has dimension n and the dimension of the Lie-algebra is k then there are $n-k$ functionally independent invariants.*
- *The functionally independent invariants are solutions to a system of k partial differential equations.*

3. Applications

In this section we will now describe applications were the construction sketched in the last section can be used to derive invariants.

3.1. Invariants for the Dichromatic Model

In many applications, for example in color image segmentation, color object recognition etc., the main interest is the physical content of the objects in the scene. Deriving features which are robust to image capturing conditions such as illumination changes, highlights, shadows and geometry changes is a crucial step in such applications [12, 1]. The interaction between lights and objects in the scene is very difficult and requires very complicated models such as Transfer Radiative Theory or Monte-Carlo simulation methods to describe. Previous studies of color invariance are, therefore, mostly based on simpler semi-empirical models such as the Dichromatic Reflection Model [14], or the model proposed by Kubelka and Munk [9], together with many additional assumptions [8, 6, 15, 3, 5]. Here we use the Dichromatic Reflection Model (see [14]) and show how to construct invariants.

It is very difficult to describe in detail what happens when light strikes a surface: some of the light will be reflected at the interface producing interface reflection, while another part will transfer through the medium undergoing absorption, scattering, and emission. The Dichromatic Reflection Model [14] describes the relation between the incoming light and the reflected light which is a mixture of the light reflected at the material surface and the light reflected from the material body. The model assumes that the light $L(x, \lambda)$ reflected from a surface can be decomposed into two additive components, an interface (specular) reflectance and a body (diffuse) reflectance under all illumination-camera geometries:

$$L(x,\lambda) = m_S(x)R_S(\lambda)E(\lambda) + m_D(x)R_D(\lambda)E(\lambda) \quad (1)$$

Here x denotes geometry changes including the angle of incidence light, the angle of remittance light and the phase angle, etc. $R_S(\lambda)$ and $R_D(\lambda)$ are the specular and diffuse reflectance respectively, and $E(\lambda)$ is the spectral power distribution of the incident light. The measured sensor values $C_n(x)$ at pixel x in the image using N filters with spectral sensitivities given by $f_1(\lambda)...f_N(\lambda)$ will be given by the following integral over the visible spectrum:

$$\begin{aligned} C_n(x) &= \int f_n(\lambda)\left[m_S(x)R_S(\lambda)E(\lambda)\right. \\ &\quad \left.+ m_D(x)R_D(\lambda)E(\lambda)\right] d\lambda \\ &= m_S(x)S_n + m_D(x)D_n \end{aligned} \quad (2)$$

Assume that two object points belong to the same material. They have therefore identical reflectance functions and the only difference are their geometrical properties. For these two neighboring pixels x_1 and x_2 and channel n we have then:

$$\begin{bmatrix} C_n(x_1) \\ C_n(x_2) \end{bmatrix} = \begin{bmatrix} m_S(x_1) & m_D(x_1) \\ m_S(x_2) & m_D(x_2) \end{bmatrix} \begin{bmatrix} S_n \\ D_n \end{bmatrix} = M \begin{bmatrix} S_n \\ D_n \end{bmatrix} \quad (3)$$

In the the framework of transformation groups we see that the matrix M operates on the vectors $(S_n\ D_n)'$. In the group theoretical approach it is now natural to construct invariants for various subgroups of the group of 2×2 matrices. A list of common subgroups is:

1. 2-D Rotations
2. Uniform scalings
3. Non-uniform scalings
4. Shears

The group of all rotations is a one-parameter group, the group of non-uniform scalings is a two parameter group and the full group is a four parameter group. Characteristic for Lie-theory is the following observation: Assume you require the transformation group to include rotations and shears. Then the properties of Lie-algebra requires you to include the scaling operations as well. Instead of creating a 2-parameter group you end up with a three-parameter group.

In most applications the camera will not only consist of one channel but of N channels. Since we separated the spectral properties and the non-spectral parameters we see that the transformation matrix M is the same for all channels. Therefore we obtain in the general case the transformation group:

$$\begin{bmatrix} C(x_1) \\ C(x_2) \end{bmatrix} = M \begin{bmatrix} S \\ D \end{bmatrix} \quad (4)$$

Here $C(x_1), C(x_2), S, D$ are N-dimensional vectors. The group now operates on the space $\left(\mathbb{R}^2\right)^N$. There are therefore $2N-k$ functionally independent invariants where k is the dimension of the Lie-algebra. For the case of RGB images and the full matrix group we have $2 \cdot 3 - 4 = 2$ invariants. A simple Maple program gives the following solutions (with $C(x_k) = (r_k, g_k, b_k)$):

$$\mathrm{f} = \_\mathrm{F1}\left(\frac{-\mathrm{g}_2\,\mathrm{b}_1 + \mathrm{b}_2\,\mathrm{g}_1}{\mathrm{r}_1\,\mathrm{g}_2 - \mathrm{r}_2\,\mathrm{g}_1}, \frac{\mathrm{b}_2\,\mathrm{r}_1 - \mathrm{r}_2\,\mathrm{b}_1}{\mathrm{r}_1\,\mathrm{g}_2 - \mathrm{r}_2\,\mathrm{g}_1}\right) \quad (5)$$

If we only start with rotations and shearing then we start with two variables but because of the Lie-structure we have $k = 3$ and there are three independent invariants:

$$\mathrm{f} = \_\mathrm{F1}(\mathrm{r}_1\,\mathrm{g}_2 - \mathrm{r}_2\,\mathrm{g}_1,\ \mathrm{b}_2\,\mathrm{r}_1 - \mathrm{r}_2\,\mathrm{b}_1,\ -\frac{-\mathrm{g}_2\,\mathrm{b}_1 + \mathrm{b}_2\,\mathrm{g}_1}{\mathrm{r}_1\,\mathrm{g}_2 - \mathrm{r}_2\,\mathrm{g}_1})$$

3.2. Kubelka-Munk Theory and Invariants

The dichromatic reflection model as described in the previous section is a quite general model but it does not take into account the physical processes once the light enters the medium. These processes include absorption, scattering, and emission. Radiative Transfer Theory [2] can be used to describe light transfer inside the medium. The Kubelka-Munk model is a special case where it is assumed that the light inside the medium is uniformly diffused and that the properties of the medium (as described by scattering and absorption coefficients) are isotropic. Under these assumptions, only two fluxes of light propagation inside the medium are enough to approximately describe the whole process. It is one of the standard tools to model the color appearance of objects and despite all its shortcomings it is still useful in many applications and has been used to derive invariants before (see [5, 4]). More detailed information about the Kubelka-Munk model can be found in [9, 13]. Here we only sketch some basic facts and derive the invariants. The Kubelka-Munk model deals only with two fluxes, one proceeding downwards (into the material and denoted by i) and the other (denoted by j) directed upwards. The downward flux is decreased by absorption (with coefficient K) and scattering (coefficient S). Similarly the upward flux j, which is reduced by absorption and scattering. The total change of the upward flux consists of two parts: the loss by absorption and scattering of the upward flux and the amount added back to the upward flux from the scattering of the downward flux. This leads to the differential equations:

$$\frac{di}{dx} = -(S+K)i + Sj \tag{6}$$

$$-\frac{dj}{dx} = -(S+K)j + Si \tag{7}$$

If the medium has optical contact with a backing of reflectance R_g, at $x = 0$ we have the following boundary condition:

$$j_0 = R_g i_0 \tag{8}$$

If the external and internal surface reflectance at the interface of the medium is denoted as r_0 and r_1, respectively, and I_0 denotes the incoming light to the interface, then the following boundary conditions can be obtained at the interface, $x = D$.

$$i_D = I_0(1-r_0) + j_D r_1 \tag{9}$$

$$I_0 R = I_0 r_0 + j_D(1-r_1) \tag{10}$$

Solving the above equations, we obtain the reflectance of the medium

$$R = r_0 + (1-r_0)(1-r_1)\cdot \frac{\left[(1-R_gR_\infty)R_\infty + (R_g - R_\infty)\mathrm{e}^{-AD}\right]}{(1-R_gR_\infty)(1-r_1R_\infty) - (R_\infty - r_1)(R_\infty - R_g)\mathrm{e}^{-AD}} \tag{11}$$

where

$$R_\infty = 1 + \frac{K}{S} - \sqrt{\frac{K^2}{S^2} + 2\frac{K}{S}} \tag{12}$$

$$A = \frac{2S(1-R_\infty^2)}{R_\infty} \tag{13}$$

A is a positive constant and if the medium is thick enough, i.e. $D \to \infty$ then

$$\tilde{R} = r_0 + \frac{(1-r_0)(1-r_1)R_\infty}{(1-r_1R_\infty)} \tag{14}$$

clearly, R_∞ is a special case of $\tilde{R}$ with interface reflections $r_0 = r_1 = 0$.

If we assume in Eq. 14 that

$$1 - r_1 R_\infty \approx 1 - r_1 \tag{15}$$

or if we have some compensation factor like a function of R_∞

$$1 - r_1 R_\infty \approx (1-r_1)\bar{g}(R_\infty) \tag{16}$$

then Eq. 14 becomes

$$\tilde{R} = r_0 + (1-r_0)g(R_\infty) \tag{17}$$

The value at pixel x under illumination $E(\lambda)$ and a camera with sensitivity function $f_n(\lambda)$ is

$$\begin{aligned} C_n(x) &= \int f_n(\lambda)E(\lambda)\left[r_0(x) + (1-r_0(x))g(R_\infty)\right]d\lambda \\ &= r_0(x)S_n + (1-r_0(x))D_n \end{aligned} \tag{18}$$

Clearly Eq. 18 has the same form as the dichromatic reflection model in Eq. 2 but the interpretation of this formula is more complicated. The external surface reflectance $r_0(x)$ depends on many factors including the incident angle of the light, the geometrical properties of the surface, the reflective index of the medium, and the polarization state of the light beam [7]. Its dependency on the wavelength can be neglected. The Kubelka-Munk coefficients K and S are the absorption and scattering coefficients of the medium along the direction in the Kubelka-Munk model. A light beam travelling inside the medium with a direction different from the direction in the Kubelka-Munk model will be absorbed and scattered more since it has to travel a longer distance. Therefore K and S depend on the direction of the light beam compared to the direction in the Kubelka-Munk model. However their ratio K/S depends only on the absorption and the scattering coefficients per unit path length of the medium. Thus R_∞ depends only on the material, but not on the direction of the light beam. Consequently, the terms S_n and D_n are independent of the geometrical properties (the incoming light and surface). A geometrical color invariant should thus be independent of the $r_0(x)$ terms.

Considering two points with coefficients $r_0^{(1)}, r_0^{(2)}$ and pixel values $C_n^{(1)}(x), C_n^{(2)}(x)$ gives the equation (compare to Eq.(4))

$$\begin{bmatrix} C^{(1)}(x) \\ C^{(2)}(x) \end{bmatrix} = \begin{bmatrix} r_0^{(1)} & 1 - r_0^{(1)} \\ r_0^{(2)} & 1 - r_0^{(2)} \end{bmatrix} \begin{bmatrix} S \\ D \end{bmatrix} = M \begin{bmatrix} S \\ D \end{bmatrix} \tag{19}$$

The problem is now that the matrices M operate on the variables S, D but that we only have access to the measurements C. After some manipulations it can be shown that the correct differential equations can be obtained in the following way: First the measurement vectors $C = \left[C^{(1)}(x), C^{(2)}(x)\right]$ have to be replaced by the transformed vectors $\widehat{C} = S^{-1}C$ with $S = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$. Then we operate with matrices of the type $L = \begin{bmatrix}x & 0\\ y & 1\end{bmatrix}$ on the transformed measurement vectors. Using the differential equations again we find now the general invariant:

$$\_F1\left(\frac{\mathrm{g}_1 - \mathrm{g}_2}{\mathrm{r}_1 - \mathrm{r}_2}, \frac{\mathrm{g}_2\,\mathrm{r}_1 - \mathrm{r}_2\,\mathrm{g}_1}{\mathrm{r}_1 - \mathrm{r}_2}, \frac{\mathrm{b}_1 - \mathrm{b}_2}{\mathrm{r}_1 - \mathrm{r}_2}, \frac{\mathrm{b}_2\,\mathrm{r}_1 - \mathrm{r}_2\,\mathrm{b}_1}{\mathrm{r}_1 - \mathrm{r}_2}\right) \tag{20}$$

A Maple worksheet used to derive the invariants is shown in the appendix.

3.3. Illumination invariants

One of the most popular problems in color image processing is "color constancy", ie. the attempt to obtain descriptions of the scene content that are independent of the illumination. In the case where the illumination spectra under consideration define a (local) Lie group it is possible to construct invariants that are constant under all illumination spectra in the Lie group. Space limitation does not allow us to describe the details of this construction here but we remark that it is possible to use Lorentz groups and the scaling group to give the space of illumination spectra a group theoretical structure. Both black-body and measured daylight sequences can be described within this framework and illumination invariants based on these methods have been described elsewhere.

4. Conclusions

We introduced the theory of Lie transformation groups and showed how they can be used to construct invariants for different reflection models. We also remarked that illumination invariants can be derived along the same lines. We showed that the theory gives both, an overview over the number of all invariants and a constructive way to find these invariants. The main contribution of this paper is not only the derivation of a number of (old and new) invariants for color vision problems but also the demonstration that there is a highly developed mathematical toolbox that allows the systematical solution to find many popular and useful invariants for color image processing. Although we cannot give a illustration of the results here (but see [16]) we can draw some general conclusions from the results obtained. Equations (5,20) show that all invariants are functions of ratios of differences. These differences will usually assume small values and the invariants will therefore be very sensitive to noise. Since we constructed all possible invariants we can conclude from the theory that all processing based on differential invariants is sensitive to noise and will probably require incorporation of additional information and/or statistical evaluation of large numbers of measurements of such invariants at different points in the image. Furthermore, since we know the form of all possible invariants we can construct the invariant with the best possible performance. An example of how to construct a new invariant is shown in the Maple program.

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A. A Maple example

```
> restart:  with(PDEtools):with(linalg):
```

The transformation matrices

```
> M := matrix(2,2,[rho[1], 1-rho[1], rho[2], 1-rho[2]]):
> S := matrix(2,2,[1,1,0,1]):
> L := evalm(inverse(S)&*M&*S);
```

$$L := \begin{bmatrix} \rho_1 - \rho_2 & 0 \\ \rho_2 & 1 \end{bmatrix}$$

Solving the differential equations

```
> eq10 :=
> subs(t=0,diff(f(exp(t)*xi[1],xi[2],exp(t)*eta[1],eta[2],
> exp(t)*zeta[1],zeta[2]),t)):
> eq1 := simplify(eq10);
> eq20 :=
> subs(t=0,diff(f(xi[1],t*xi[1]+xi[2],eta[1],t*eta[1]+eta[2],zeta[1],
> t*zeta[1]+zeta[2]),t)):
> eq2 := simplify(eq20);
```

$$\mathit{eq1} := \mathrm{D}_1(f)(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)\,\xi_1 + \mathrm{D}_3(f)(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)\,\eta_1 + \mathrm{D}_5(f)(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)\,\zeta_1$$

$$\mathit{eq2} := \mathrm{D}_2(f)(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)\,\xi_1 + \mathrm{D}_4(f)(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)\,\eta_1 + \mathrm{D}_6(f)(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)\,\zeta_1$$

```
> inv0 := pdsolve({eq1,eq2},[f]);
> rgbinv :=
> map(simplify,subs(xi[1]=r[1]-r[2],xi[2]=r[2],eta[1]=g[1]-g[2],
> eta[2]=g[2],zeta[1]=b[1]-b[2],zeta[2]=b[2],inv0));
> rgbinvfun  :=
> unapply(F(1/(r1-r2)*(g1-g2),(g2*r1-r2*g1)/(r1-r2),1/(r1-r2)*(b1-b2),
> (b2*r1-r2*b1)/(r1-r2)),[r1,r2,g1,g2,b1,b2]):
```

$$\mathit{inv0} := \{\mathrm{f}(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2) = \_\mathrm{F1}(\frac{\eta_1}{\xi_1}, \frac{\eta_2\,\xi_1 - \eta_1\,\xi_2}{\xi_1}, \frac{\zeta_1}{\xi_1}, \frac{\zeta_2\,\xi_1 - \zeta_1\,\xi_2}{\xi_1})\}$$

$$\mathit{rgbinv} := \{\mathrm{f}(\mathrm{r}_1 - \mathrm{r}_2, \mathrm{r}_2, \mathrm{g}_1 - \mathrm{g}_2, \mathrm{g}_2, \mathrm{b}_1 - \mathrm{b}_2, \mathrm{b}_2) = \_\mathrm{F1}(\frac{\mathrm{g}_1 - \mathrm{g}_2}{\mathrm{r}_1 - \mathrm{r}_2}, \frac{\mathrm{g}_2\,\mathrm{r}_1 - \mathrm{r}_2\,\mathrm{g}_1}{\mathrm{r}_1 - \mathrm{r}_2}, \frac{\mathrm{b}_1 - \mathrm{b}_2}{\mathrm{r}_1 - \mathrm{r}_2}, \frac{\mathrm{b}_2\,\mathrm{r}_1 - \mathrm{r}_2\,\mathrm{b}_1}{\mathrm{r}_1 - \mathrm{r}_2})\}$$

Test the solution

```
> TT := matrix(2,3,[r1,g1,b1,r2,g2,b2]):TM := matrix(2,2,[x,1-x,y,1-y]):
> SS := evalm(TM&*TT):
> map(simplify,rgbinvfun(SS[1,1],SS[2,1],SS[1,2],SS[2,2],SS[1,3],SS[2,3]));
```

$$\mathrm{F}(\frac{g1 - g2}{r1 - r2}, \frac{g2\;r1 - r2\;g1}{r1 - r2}, \frac{b1 - b2}{r1 - r2}, \frac{b2\;r1 - r2\;b1}{r1 - r2})$$

Example to show how to construct a new invariant:

```
> ifun1 := (r1,r2,g1,g2,b1,b2)->1/(r1-r2)*(g1-g2):
> ifun2 := (r1,r2,g1,g2,b1,b2)->(g2*r1-r2*g1)/(r1-r2):
> ifun3 := (r1,r2,g1,g2,b1,b2)->1/(r1-r2)*(b1-b2):
> ifun4 := (r1,r2,g1,g2,b1,b2)->(b2*r1-r2*b1)/(r1-r2):
> simplify(ifun2(r1,r2,g1,g2,b1,b2)*ifun3(r1,r2,g1,g2,b1,b2)-ifun4(r1,r
> 2,g1,g2,b1,b2)*ifun1(r1,r2,g1,g2,b1,b2));
```

$$\frac{g2\;b1 - b2\;g1}{r1 - r2}$$