Theoretical Analysis of Subtractive Color Mixture Characteristics

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Abstract

In this paper, on the assumption of the Lambert-Beer model for subtractive color mixture, several theorems regarding the stimulus values of colors obtained by subtractive color mixture are proved. The results of this paper will contribute to a systematization of subtractive color mixture.

Introduction

In this paper, we discuss the characteristics of subtractive color mixture.[1-5]

On the assumption of the Lambert-Beer model for subtractive color mixture, several theorems regarding the stimulus values of colors obtained by subtractive color mixture are proved. Generally, in subtractive color mixture, the resultant stimulus values are not uniquely determined from component stimulus values. Hence, analysis regarding the maximum and the minimum stimulus values are very important to understand the possible ranges of stimulus values of subtractive color mixture. On which parameter the maximum stimulus value of a subtractive color mixture depends? Which parameters can be related to illuminant coordinate decomposition? This paper provides theorems about these issues. There have not been studies about these issues before.

The results of this paper will contribute to a systematization of subtractive color mixture.

Subtractive Color Mixtureand Model

Additive color mixture results from the mixture of colored light, while subtractive color mixture results from superimposing absorption media, such as coloring materials and color filters. In this paper, the Lambert-Beer model[4] is assumed for the subtractive color mixture model. Denote the spectral transmittances for cyan (*C*), magenta (*M*), yellow (*Y*) dye as $T_C(\lambda)$, $T_M(\lambda)$, $T_Y(\lambda)$, and the density for each primary as *c, m, y*, respectively. The spectral transmittance $T(\lambda)$ for colorant layers of *C, M, Y* is modeled as follows:

$$\begin{aligned} T(\lambda) &= \{T_C(\lambda)\}^c \{T_M(\lambda)\}^m \{T_Y(\lambda)\}^y \\ &= \rho_1(\lambda)\rho_2(\lambda)\rho_3(\lambda), \end{aligned} \tag{1}$$

where

$$\begin{aligned} &\rho_1(\lambda) = \{T_C(\lambda)\}^c, \\ &\rho_2(\lambda) = \{T_M(\lambda)\}^m, \\ &\rho_3(\lambda) = \{T_Y(\lambda)\}^y \\ &0 \le \rho_i(\lambda) \le 1 \ (i = 1,2,3) \end{aligned}$$

The tri-stimulus values (*X, Y, Z*) are calculated as follows:

$$\begin{aligned} X &= \int T(\lambda)S(\lambda)\bar{x}(\lambda)d\lambda, \\ Y &= \int T(\lambda)S(\lambda)\bar{y}(\lambda)d\lambda, \\ Z &= \int T(\lambda)S(\lambda)\bar{z}(\lambda)d\lambda, \end{aligned} \tag{2}$$

where $S(\lambda)$: spectral power distribution of illuminant, $\bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda)$: color matching functions.

Theoretical Analysis

In this theoretical analysis, the ideal color model[1] is assumed, and $\rho_i(\lambda)$ (I = 1,2,…,n) take only 1 or 0 values. Hereafter, $n(2 \le n)$ colorants means a layer composed of n colorants. For the proof of general $n(2 \le n)$ colorants case, the mathematical induction is employed. First, the mathematical relation of $n = 2$ colorants case is proved, and after that $n(2 \le n)$ colorants case is proved under the assumption of the mathematical relation of the $(n - 1)$ colorants case.

[Notation]

max $\left[X_1, X_2, \cdots, X_n\right]$: maximum value among $X_i (i = 1,2,\ldots,n)$,

min $\left[X_1, X_2, \cdots, X_n\right]$: minimum value among $X_i (i = 1,2,\ldots,n)$,

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle = \int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda)\bar{x}(\lambda)d\lambda,$$

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\max} = \max_{\rho_i(i=1,2,\cdots,n)} \left[\int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda)\bar{x}(\lambda)d\lambda \right]$$

: maximum value of $\int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda)\bar{x}(\lambda)d\lambda$

parameterized by $\rho_i(\lambda)\ (i = 1,2,\cdots,n)$,

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\min} = \min_{\rho_i(i=1,2,\cdots,n)} \left[\int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda)\bar{x}(\lambda)d\lambda \right]$$

: minimum value of $\int \left(\prod_{i=1}^{n} \rho_i(\lambda) \right) S(\lambda)\bar{x}(\lambda)d\lambda$

parameterized by $\rho_i(\lambda)\ (i = 1, 2, \cdots, n)$,

where

$\rho_i(\lambda) \quad (i = 1,2,\cdots,n)$ for $\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\max}$ and $\rho_i(\lambda)$

$(i = 1,2,\cdots,n)$ for $\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\min}$ are optimized

independently.

[Definition]

Stimulus Values

Define a stimulus value for each colorant layer as follows:

$$X_i = \langle\!\langle \rho_i(\lambda) \rangle\!\rangle \quad (i = 1,2,\cdots,n). \tag{3}$$

Equation (3) is the defining constraint of the whole problem, and X_i ($i = 1,2,\ldots,n$), are inputs that define the constraint.

Relations in the Spectral Transmittance of n Colorants

- Included relation in the transmittance of n colorants

The included relations of transparent bands are defined as the cases that a wavelength region of ρ = 1 of a colorant is included in wavelength regions of ρ = 1 of other colorants larger than the region, as shown in Figure 1 as an example. Figure 1 shows the transmittance composed by n colorants and the index i ($1 \leq i \leq n$) region of ρ = 1 is included by the smaller index number of regions of ρ = 1. The solid lines indicate the results of the superimposing the colorants.

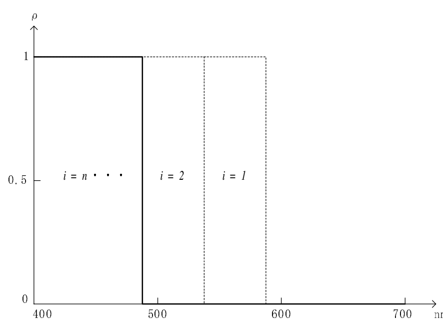

Figure 1. Example of included relations.

- Separated relation in the transmittance of n colorants

The separated relations of absorption bands are defined as the cases that a wavelength region of ρ = 0 of a colorant is separated with wavelength regions of ρ = 0 of other colorants as shown in Figure 2 as an example. Figure 2 shows the transmittance composed by n colorants, and the absorption band i ($i = 1,2,\ldots,n$) corresponds to the absorption band for each colorant i ($i = 1,2,\ldots,n$).. The solid lines indicate the results of the superimposing the colorants.

The same relations consist for combinations of various types of the ideal color model other than the examples shown in Figure 1 and Figure 2.

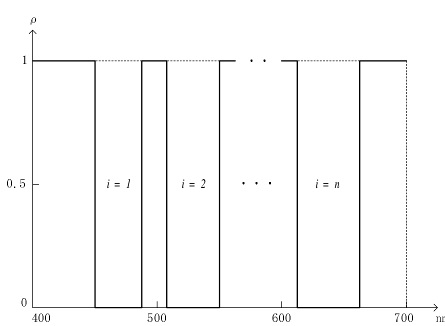

Figure 2. Example of separated relations.

[A priori Condition]

Spectral power distribution of illuminant S(λ) and color matching functions $\bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda)$ are fixed. The values of $X_i (i = 1,2,\ldots,n)$ are given a priori. The priori conditions are used for theorems following.

Theorems 1, 2 are about which parameters the maximum stimulus value of a subtractive color mixture corresponds. Theorems 1, 2 are useful to determine the maximum stimulus value of subtractive color mixture under the assumption that the stimulus value of each colorant is a priori given.

Theorem 1 is for $n = 2$ colorants case, and theorem 2 is for the mathematical relation of $n\,(2 < n)$ colorants case under the assumption of the mathematical relation of $(n - 1)$ colorants case.

[Theorem 1]

For a priori given X_1, X_2, the following mathematical relation is valid:

$$\langle \rho_1(\lambda)\rho_2(\lambda) \rangle_{\max} = \min[X_1, X_2]\,. \tag{4}$$

Proof

Consider the Case of $X_1 < X_2$.

The left side of Eq.(4) is maximized when $\rho_1(\lambda)$ is included in $\rho_2(\lambda)$ (included relation of transparent band) on the reason that $\rho_1(\lambda)\bullet\rho_2(\lambda)$ takes the maximum ratio value of 1 in the wavelength range. Since in this case $\rho_1(\lambda)\bullet\rho_2(\lambda) = \rho_1(\lambda)$ and solving the left side of Eq.(4) we have the right side of Eq.(4). The included relation always exists for any values of X_1, X_2, because $\rho_1(\lambda)$ can exists as a part of $\rho_2(\lambda)$ in the case of $X_1 < X_2$.

In the case of $X_1 \leq X_2$, the proof is provided in the same way.

[Theorem 2]

For a priori given $X_1, X_2, \ldots, X_n$, the following mathematical relation is valid for n colorants layer :

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\max} = \min[X_1, X_2 \cdots, X_n]\,. \tag{5}$$

Proof

By using the notation of

$$\rho_1(\lambda)\rho_2(\lambda)\cdots\rho_{n-1}(\lambda) = \rho^*_{n-1}(\lambda)\ ,\ \ \min[X_1, X_2, \cdots, X_{n-1}] = X^*_{n-1}$$

and from this theorem for $(n - 1)$ colorants, the following relation is derived:

$$X^*_{n-1} = \langle \rho^*_{n-1}(\lambda) \rangle_{\max} = \min[X_1, X_2, \cdots, X_{n-1}]. \tag{6}$$

Based on Eq.(6) and theorem 1, the following relation is derived,

$$\begin{aligned}\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\max} &= \langle \rho^*_{n-1}(\lambda)\rho_n(\lambda) \rangle_{\max} = \min[X^*_{n-1}, X_n] \\ &= \min[X_1, X_2, \cdots, X_n]\,.\end{aligned} \tag{7}$$

The theorem is proved.

Mathematical induction based on theorem1 and theorem 2 proves the general case of $2 \leq n$.

Theorems 3, 4 are about which parameters can be related to illuminant coordinate decomposition in subtractive color mixture. Theorems 3, 4 are useful to understand the components of an illuminant, under the assumption that the stimulus value of each colorant is a priori given.

[Theorem 3]

X_0 indicates just the illuminant X value.

If $X_0 < X_1 + X_2$, then X_0 is represented as follows:

$$X_0 = X_1 + X_2 - \langle \rho_1(\lambda)\rho_2(\lambda) \rangle_{\min}\,. \tag{8}$$

Proof

The assumption of $X_0 < X_1 + X_2$ is explained. On the condition of $X_0 = X_1 + X_2$, there exist $\rho_1(\lambda)$, $\rho_2(\lambda)$ whose transparent bands completely fill up all the wavelength range by the combinations without overlapping and satisfies $X_0 = X_1 + X_2$. Starting from $\rho_1(\lambda)$, $\rho_2(\lambda)$ satisfying $X_0 = X_1 + X_2$, consider the case of $X_0 < X_1 + X_2$. On the condition of $X_0 < X_1 + X_2$, absorption bands of $\rho_1(\lambda)$, $\rho_2(\lambda)$ should be decreased for the increase of $X_1 + X_2$, and the absorption bands can be separated each other in $\rho_1(\lambda)\rho_2(\lambda)$, because, there can exist slates of $\rho_1(\lambda)=1$, $\rho_2(\lambda)=1$ at adjacent boundary by adequate decrements of each absorption band width.

As explained above, the assumption of $X_0 < X_1 + X_2$ corresponds to that there exist $\rho_1(\lambda)$ and $\rho_2(\lambda)$ whose absorption bands are separated in $\rho_1(\lambda)\rho_2(\lambda)$ for given values of X_0, X_1, X_2.

A separated relation between $\rho_1(\lambda)$ absorption bands (where $\rho_1(\lambda)=0$) and $\rho_2(\lambda)$ absorption bands (where $\rho_2(\lambda)=0$) minimizes $<\rho_1(\lambda)\rho_2(\lambda)>$in Eq.(8), because $\rho_1(\lambda)\bullet\rho_2(\lambda)$ takes the maximum ratio of 0 value in the wavelength range for given X_1 and X_2. The assumption of $X_0 < X_1 + X_2$ enables the separated relation between $\rho_1(\lambda)$ absorption bands and $\rho_2(\lambda)$ absorption bands. Using these relations, the following equation is derived :

$$\begin{aligned}&X_1 + X_2 - \langle \rho_1\langle\lambda\rangle\rho_2(\lambda) \rangle_{\min} \\ &= (X_0 - (\text{absorption}\quad \text{by}\ \ \rho_1(\lambda))) \\ &+ (X_0 - (\text{absorption}\quad \text{by}\ \ \rho_2(\lambda))) \\ &- (X_0 - (\text{absorption}\quad \text{by}\ \ \rho_1(\lambda)\ \text{and}\ \ \rho_1(\lambda))) \\ &= X_0,\end{aligned} \tag{9}$$

where

(absorption by $\rho_i(\lambda)$): X value absorbed by $\rho_l(\lambda)$ (i = 1 or 2).

(absorption by $\rho_1(\lambda)$ and $\rho_2(\lambda)$):X value absorbed by $\rho_1(\lambda)$ (i=1,2),

(absorption by $\rho_1(\lambda)$ and $\rho_2(\lambda)$)

=(absorption by $\rho_1(\lambda)$)+(absorption by $\rho_2(\lambda)$), under the separated relation.

The theorem is proved.

[Theorem 4]

If $(n-1)X_0 < \sum_{i=1}^{n} X_i$, then X_0 is represented as follows:

$$X_0 = \left(\sum_{i=1}^{n} X_i - \left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\min} \right) / (n-1). \tag{10}$$

Proof

The assumption of

$$(n-1)X_0 < \sum_{i=1}^{n} X_i$$

is explained. The explanation is the extension of the explanation of Theorem 3 on the same theoretical framework. On the condition of

$$(n-1)X_0 = \sum_{i=1}^{n} X_i$$

there exist $\rho_1(\lambda),\rho_2(\lambda),\ldots,\rho_n(\lambda)$ whose transparent bands completely fill up all the wavelength range (n – 1) times by the combinations without overlapping and satisfies

$$(n-1)X_0 = \sum_{i=1}^{n} Xi .$$

Starting from $\rho_1(\lambda),\rho_2(\lambda),\ldots,\rho_n(\lambda)$ satisfying the relation of

$$(n-1)X_0 = \sum_{i=1}^{n} X_i,$$

consider the case of

$$(n-1)X_0 < \sum_{i=1}^{n} X_i.$$

On the condition, absorption bands of $\rho_1(\lambda),\rho_2(\lambda),\ldots,\rho_n(\lambda)$ should be decreased for the increase of

$$\sum_{i=1}^{n} X_i ,$$

and the absorption bands can be separated each other in $\rho_1(\lambda)\rho_2(\lambda)\ldots\rho_n(\lambda)$, because, there can exist slates of $\rho_1(\lambda)$ = 1, $\rho_2(\lambda)$ = 1, …,$\rho_n(\lambda)$=1 at adjacent boundary by adequate decrements of each absorption band width.

As explained above, the assumption of

$$(n-1)X_0 < \sum_{i=1}^{n} X_i$$

corresponds to that there exist $\rho_1(\lambda),\rho_2(\lambda),\ldots,\rho_n(\lambda)$ whose absorption bands are separated in $\rho_1(\lambda)\rho_2(\lambda)\ldots\rho_n(\lambda)$ for given values of X_0, $X_1,\ldots,X_n$. For the same $\rho_1(\lambda),\rho_2(\lambda),\ldots,\rho_n(\lambda)$, the absorption bands between $\rho_n(\lambda)$ and

$$\prod_{i=1}^{n-1} \rho_i(\lambda)$$

are also separated that under the assumption there exist $\rho_n(\lambda)$ and

$$\prod_{i=1}^{n-1} \rho_i(\lambda)$$

whose absorption bands are separated.

Assume that Eq.(10) is valid for (n – 1) colorants, Eq.(10) for (n – 1) colorants is converted as follows:

$$(n-2)X_0 = \left(\sum_{i=1}^{n-1} X_i - \left\langle \prod_{i=1}^{n-1} \rho_i(\lambda) \right\rangle_{\min} \right). \tag{11}$$

The minimization of

$$\left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle$$

is attained when $\rho_n(\lambda)$ is separated from

$$\prod_{i=1}^{n-1} \rho_i(\lambda)$$

(separated relation of absorption bands). The separation is assured from the assumption. On the separation, the relation of

$$\left(\left\langle \prod_{i=1}^{n-1} \rho_i(\lambda) \right\rangle_{\min} - \left(absorption \quad by \;\; \rho_n \right) \right) = \left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\min}$$

is valid. And the subtraction X_0-(absorption by $\rho_n(\lambda)$) corresponds to X_n that the following equation is derived using Eq.(11):

$$(n-1)X_0 = X_0 + \left(\sum_{i=1}^{n-1} X_i - \left\langle \prod_{i=1}^{n-1} \rho_i(\lambda) \right\rangle \right)_{\min}$$

$$= \left(X_0 - \left(absorption \quad by \ \rho_n(\lambda)\right)\right) + \sum_{i=1}^{n-1} X_i$$

$$- \left(\left\langle \prod_{i=1}^{n-1} \rho_i(\lambda) \right\rangle - \left(absorption \quad by \ \rho_n(\lambda)\right)\right)$$

$$= \sum_{i=1}^{n} X_i - \left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\min} . \tag{12}$$

In Eq. (12), an addition and a corresponding subtraction of the (*absorption by* $\rho_1(\lambda)$) terms whose summation equals to 0.0 are included in the second equation from the deformation of the first equation and the final form of Eq.(12) is derived. Dividing the both side of Eq.(12) by $n - 1$, the equation for n colorants layer is derived as follows:

$$X_0 = \left(\sum_{i=1}^{n} X_i - \left\langle \prod_{i=1}^{n} \rho_i(\lambda) \right\rangle_{\min} \right) / (n-1) . \tag{13}$$

Mathematical induction based on theorem 3 and theorem 4 proves the general case of $2 \leq n$.

In theorems 1-4, the relation about the *X* stimulus values are proved. The same relation for *Y*, *Z* stimulus values are proved in the same way.

Conclusions

In this paper, we have discussed about the characteristics of subtractive color mixture. Analysis regarding the maximum and the minimum stimulus values are very important to understand the possible ranges of stimulus values of subtractive color mixture.

The results of this paper will contribute to a systematization of subtractive color mixture. In this paper, characteristics along a single axis of tristimulus values were discussed. Hereafter, characteristics along three dimensional directions in the tristimulus space will be discussed.

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Biography

Dr. Nobuhito Matsushiro received his Ph.D degree from the University of Electro. -Communications, Tokyo, Japan, in 1996, related to information theory. He is a visiting scientist at Munsell Color Science Laboratory, Center for Imaging Science, Rochester Institute of Technology.

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