Are *a priori* metrics in colorimetry meaningful?

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Abstract

"Are a priori *metrics in colorimetry meaningful?" is a question that is rarely asked. Yet the choice of metric in the form of an inner product is of crucial importance in basic colorimetry. Is there an* a priori *unique choice? The answer is* no. *One needs to opt for some choice of inner product on rational grounds that may differ from case to case. I discuss possible alternatives.*

Introduction

Assuming Graßmann's Laws[1], "color space" is a linear space, or, more precisely, a convex cone in a three-dimensional real vector space. Elements of the space are "colors", which are equivalent classes of radiant powerspectra called "metamers", a term borrowed from chemistry by Ostwald[2]. The space of spectra is a convex cone in an infinitely dimensional linear topological space. It is a Hausdorff space, though not a Hilbert space, for there is no natural inner product. This motivates my title: it is common enough to treat the space of spectra as a Hilbert space. Is there an *a priori* reason why this might make sense? If not, what are the consequences, and are there reasonable alternatives?

Radiant power spectra

Since Newton[3] radiant beams are usually described in terms of a spectral basis. One specifies radiant power density on wavelength basis. It is of some importance to appreciate that this is just one of of infinitely many, mutually equivalent bases. Thus it is strictly nonsense to say (as modern textbooks still do) that general beams of radiation are "composed of monochromatic components". This is about as meaningful as to say that four is "two plus two" as opposed to "one plus three". But for a statement to "not be false" is not the same as for it to be "right". Of the infinitely many possible bases at least some have an obvious relevance, for instance you might specify photon number density on photon energy basis.

This is all trivial, of course. The problem (as signaled in the title) is that it is common practice to *implicitly* introduce an inner product, namely the "obvious" (Cartesian) inner product in the radiant power on wavelength basis representation. This goes usually unnoticed, although there are infinitely many choices available for an inner product. One merely needs a symmetric function that is positive definite and linear in the first argument. With such a range of choices one surely needs a *reason* for a choice! It makes a difference.

Why is this rarely noticed? The reason is that one apparently believes that matrices can be transposed with perfect freedom. They can't. To transpose a matrix is to switch to the adjoint of a linear map. In order to define a transpose one needs the dual vector space. In order to define a dual vector space one needs an inner product.

Many standard methods in colorimetry implicitly assume the existence of the dual space and the adjoint. A key example is Cohen's[4] "Matrix R". This is a tool with powerful applications, so its popularity is easy to understand. However, blindly computing Cohen's Matrix R involves the (to many apparently implicit) choice of an inner product. Perhaps unfortunately, the results depend on this choice. The "fundamental" and "black" components of a generic beam will turn out to be different for different choices of the inner product.

Of course, this means that there is little "fundamental" about fundamental spectra. This implies that the choice of metric (or inner product) is of *crucial importance*.

The quest for a canonical basis

The "standard basis" for the space of radiant spectra is Newton's spectrum, understood as radiant power density on wavelength basis. Maxwell[5] was the first to "gauge the spectrum" in the mid nineteenth century, that is to say, to measure a projection operator into three-dimensional "color space". This involves the essentially arbitrary choice of three fiducial spectra as a basis. The standard CIE color matching functions define exactly such a projection operator. The color coordinates are simply the weights by which the fiducial spectra are added. A picture of color space is obtained by plotting these weights in terms of a Cartesian system. Such a picture is meaningful if you view it *modulo* arbitrary linear transformations. Thus, it is not meaningful to compare lengths in different directions, nor to consider angles. Since people tend to forget this (it is not "natural" to view a picture *modulo* arbitrary linear deformations) it tends to be misleading.

This is where Cohen's Matrix R comes to the rescue. It is essentially a singular values decomposition, which lets you construct a nice orthonormal basis of "fundamental space". Now the picture of color space becomes a "true" picture in which Euclidean measures are meaningful.

However, because the result is dependent upon the choice of inner product, it should be understood *modulo* any change of inner product. This means that you can pick any three (independent, and not in the kernel of the projection) radiant spectra and construct an inner product that makes them into an orthonormal basis. Thus the progress is actually nil. If virtually any three-dimensional subspace can be promoted to "fundamental" space, there is little "fundamental" about it!

The only true invariant is the kernel of the projection. It is well defined in the absence of an inner product.

A priori choice of metric on a rational basis

Is it possible to make a choice (preferably a unique choice) on a rational basis? The reason should be found in physics, since we are dealing with the space of radiant beams.

Perhaps one could frame a story that would make Cohen's implicit choice sound like a "natural" choice. No doubt one could frame other stories that would yield a different choice though. It depends upon the application. If the detector is of the thermal kind radiant power is important, if it is a photon counter photon number density would be preferable. In most applications photon energy would be more important than wavelength in vacuum. And so forth.

Although certainly not ruled out, so far I have seen no particular choice that would surely convince everybody.

Alternatives to the choice of a metric

Since the *a priori* choice of metric on rational grounds is not an easy matter, one might look for alternatives. These need not necessarily be sought for in physics. For instance, because you may freely choose fundamental space, you might select three fiducial beams and promote them to the preferred orthonormal basis of fundamental space. Of course one would need a convincing story to "sell" such a choice.

Example of a rational choice

One way to arrive at a unique rational choice is as the optimum of some desirable property. The arbitrariness is then shifted to the choice of desirable property. It is often easier to construct a compelling story for that.

In the case of colorimetry *per se* it is not obvious how to proceed. The case of the (formal) object colors is easier because one has strong non-linear constraints there. A (formal) object color is defined as a radiant beam whose radiant spectral power density is less than or equal to the radiant spectral power density of some fiducial beam denoted the "illuminant". In the space of radiant beams these colors are confined to a hyper-cuboid with edge lengths given by the radiant spectral power density of the illuminant. The projection in color space is a convex body (the "color solid") with central symmetry (these properties are inherited from the hypercuboid), that is smooth except for two ridges that connect two conical singular points. One conical point is at the origin (the tangent cone being the spectrum cone), the other at the "white point", that is the color of the illuminant. These properties are geometrically obvious, they were formally proven by Schrödinger[6] in the nineteentwenties.

Of course the white point depends upon the illuminant. Even for the same white point the shape of the color solid depends upon the spectrum of the illuminant, thus beams metameric to the illuminant yield distinct color solids.

A basis of object colors is just a choice of three points within the color solid. All colors that may be produced by combining the basis colors with weight between zero and one lie in a parallelepiped. In order to "capture" as many colors as possible, this parallelepiped should exhaust as much of the volume of the color solid as possible. Here is an optimum principle. How useful is it?

There are a number of issues of relevance here. One is that the choice of illuminant is still arbitrary. Although this may be granted, there is a good story (involving the evolution of the human species) to tell that would imply "natural daylight". This limits the choice tremendously. Your favorite natural daylight spectrum will do fine. The other issue involves the nature of the optimum.

It is geometrically obvious that the major diagonal of the parallelepiped should be the segment that connects the origin to the white point, and that the basis colors should be on the boundary of the color solid. More specifically, intuitively the optimum is reached when the tangent plane at one basis color is parallel to the plane spanned by the other two. The nature of points on the boundary of the color solid is well known: the radiant spectral power density equals that of the illuminant or is zero, with no more than two transitions throughout the spectrum. This implies that the basis colors are derived as "parts of the illuminant" (a notion suggested by Schopenhauer[7] in the first half of the nineteenth century), obtained by cutting the spectrum at two or three spectral locations.

The partition has to be found numerically, an exhaustive search yields a single solution with two spectral cuts. Thus the basis colors are short, medium and long, mutually abutting wavelength regions. I will (this is merely colorful language, not an attempt to sneak in *qualia*) refer to these as the blue, green and red parts of the illuminant.

Thus we arrive at a rational choice. It involves the choice of illuminant, defendable on the basis of ecological and evolutionary arguments, and the notion that the fundamental colors should maximize the object color gamut. As a final step the parallelepiped is promoted to the unit cube ("RGB cube"), thereby fixing the metric.

How useful is the example?

The example turns out to be very useful, thus bearing out Schopenhauer's hunch. The RGB cube exhausts most of the color solid and may be substituted for it in many applications. Conversely, the color solid is very similar to a slightly inflated unit cube in this metric. For instance, the (of course periodic) sequence **R**–**Y** $(= \mathrm{R}+\mathrm{G})$–**G**–**C** $(= \mathrm{G}+\mathrm{B})$–**B**–**M** $(= \mathrm{B}+\mathrm{R})$ appears as a (non-planar) hexagon with almost identical edge lengths, thus yielding a virtually perfect "color circle", and so forth.

For applications in computer graphics and imaging this may prove to be the representation of choice. It is transparent to the intuition, and leads to simple algorithms.

Conclusion

Are *a priori* metrics in colorimetry meaningful? Yes, they can be, although they usually aren't, because introduced implicitly, without rational motivation.

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Author Biography

Jan Koenderink (1943) graduated in 1972 in physics, mathematics, and astronomy at Utrecht University. From the late 1970s he held a chair "The Physics of Man" at Utrecht University till his retirement in 2008. He is connected with KU Leuven and TU Delft. He is a member of the Dutch Royal Society of Arts and Sciences (KNAW) and received a honorific doctorate in medicine from Leuven University. Current interests include the mathematics and psychophysics of space and form in vision, including applications in art and design.