On Curvature of Color Spaces and its Implications

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Abstract

In this paper we discuss the role of curvature in the context of color spaces. Curvature is a differential geometric property of color spaces that has attracted less attention than other properties like the metric or geodesics. In this paper we argue that the curvature of a color space is important since curvature properties are essential in the construction of color coordinate systems. Only color spaces with negative or zero curvature everywhere allow the construction of Munsell-like coordinates with geodesics, shortest paths between two colors, that never intersect. In differential geometry such coordinate systems are known as Riemann coordinates and they are generalizations of the well-known polar coordinates.

We investigate the properties of two measurement sets of just-noticeable-difference (jnd) ellipses and color coordinate systems constructed from them. We illustrate the role of curvature by investigating Riemann normal coordinates in CIELUV and CIELAB spaces. An algorithsm is also shown to build multi-patch Riemann coordinates for spaces with the positive curvature.

Introduction

The fact that a color space is a Riemann space, or a curved space, rather than a Euclidean space was first understood by Helmholtz, whose well known line element is the first effort in vision research to define a Riemann metric tensor for the color space characterizing human color vision. An historical account of related studies of color spaces and their foundations in Riemann geometry can be found in [14] and [13].

A major difference between a curved space and a Euclidean space is that cartesian coordinates are only meaningful in local neighborhoods of points. Therefore it is usually very hard to characterize quantitative properties and relationships among whole distributions of color stimuli. An example is the difficulty in investigating large color differences.

Riemannian geometry provides powerful tools with which one can use to construct coordinate systems in a color space that are similar to a coordinate system in a Euclidean space. In particular, the color-difference between two color stimuli can be measured by the geodesic distance between them. As an analogy to the Munsell system, the surfaces of constant brightness correspond to the surfaces with constant geodesic distance from the origin, the lines of constant hues are geodesics starting from the achromatic origin on the constant brightness surface and the lines of constant chroma are the closed curves with constant geodesic distance from the origin on the constant brightness surface (for more information see [14]).

This Munsell-like coordinate system in a color space is known as the Riemann normal coordinate system which is a generalization of the polar system in a Euclidean space. This coordinate system has many favorable properties and plays an important role in various applications. Examples are the construction of isometry or color-difference-preserving maps for uniform color spaces, color-weak correction and color reproduction (see [4], [5] and [10] for some examples).

It would be of great advantage if one could construct such a Riemann normal coordinate system in all color spaces, but this is unfortunately not always possible. In fact, the existence of such a coordinate system depends on one of the most important properties of a color space as a Riemann manifold, the Riemann curvature. The Riemann curvature tensor describes the bending of the space, that is how much and in what way the Riemann space deviates from a flat space.

It is known as an invariant of a Riemann space under isometries or distance preserving maps. Therefore the curvature of a color space is very important for both theoretical and practical reasons, especially when one transforms one color space into another. A simple first test to check color-difference preservation is to see if the curvature is preserved. e.g. since a uniform color space is isometrc to the Euclidean space, the curvature in a uniform color space should be zero everywhere.

However, although the metric is well studied in colorimetry, the curvature issue seems to have been regarded as a purely theoretical subject and has not attracted sufficient attention until now. An interesting illustration of the importance of curvature is the conclusion that a chromaticity plane cannot be embedded into 2D Euclidean space because of its nonzero curvature (see [12] and [9]). Curvature is also used in [14] to show that the Stiles line element model is not compatible with MacAdam's ellipses since they have different signs in the xy chromaticity diagram.

In fact, the discrepancy between the Stiles line element and the MacAdam ellipses has much more serious implications than expected. In particular, the curvature plays such an important role in a color space that it is vital for the existence of Riemann normal coordinates.

Radiating straight lines emerging from the origin in a Euclidean space will never intersect each other but in a general Riemann space geodesics may intersect each other. A well-known example is the sphere where the great-circles are the geodesics and infinitely many of them intersect at the two poles of the sphere. In that case we may not be able to obtain well defined coordinates and lines of constant hue or constant chroma from geodesics. Thus the possibility to draw geodesics through the whole chromaticity diagram means that the later can be covered by a single Riemann normal coordinate neighborhood, which requires that the whole diagram has negative or zero curvature.

Therefore it is not always possible to have a global Munsell-like or Riemann normal coordinate system for a color space. In other words, usually such a Riemann normal coordinate system will only exists locally, i.e., at certain neighborhood of a point in the space. Consequently, one may be able to uniformize, or to build a color-difference-preserving map, around a neighborhood of every point in such a color space using the local Riemann normal coordinates, but usually this can not be extended to a global uniformization or color-difference preservation.

In this paper we will show how curvature effects the properties of the color spaces. The curvatures of two sets of measurements of threshold ellipses will be calculated for the CIELUV and CIELAB spaces. We will present the intersection problem of geodesics and show that the problem originates in the positive curvature. In such a case, the Riemann normal coordinate system does not exists. We show that by applying smoothing one can reduce fluctuation of the curvature, but it also changes the shapes of jnd ellipses, and therefore we need a better interpolation strategy.

Finally, we also show that in a case where we have a positive curvature that we can not circumvent, we will use the comparison theory of Riemann manifolds to find the injective radius or the minimal distance before the geodesics intersect. Using this information, one is able to build a multi-patch geodesics coordinate system for the whole color space. Moreover, a combination of smoothing with the multi-patch strategy is also discussed in order to reduce integral error and for fast implementation using parallel processing.

Curvature of color spaces

We know that the squared length of a vector in a Euclidean space is measured by the self-inner product of the vector. An n-D Riemann space is a space in which the distance can be only measured locally, or the length of a very small vector $(dx^1,\dots,dx^n)'$ around a point $x=(x^1,\dots,x^n)$ can be measured using an extended inner product as follows or a quadratic form defined by an $n\times n$ matrix $G(x)=(g_{ij}(x))$ called a metric tensor.[2]

$$ds^2=\sum_i g_{ij}dx^i dx^j$$

This is called a line element in color science. In a color space, the Riemann metric is defined by the jnd ellipses.

On a Riemann space $(\mathbb{R}^n, G)$ with the metric tensor G, an extension of derivative in Euclidean space known as covariant derivative ∇ is defined in the following way:

$$\nabla_{\partial_i}\partial_j=\sum_k \Gamma^k_{ij}\partial_k, \quad (1)$$

where the $\partial_i := \frac{\partial}{\partial x^i}$, Γ^k_{ij} are known as Christoffel symbols. It is known that there is a unique covariant derivative called compatible with the metric G, where the Christoffel symbol is calculated from G as follows.[2]

$$\Gamma^k_{ij}=\frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i}+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right) \quad (2)$$

where "g^{kl}" are entries of the inverse matrix of G. It is known that the curvedness of a surface in 3D Euclidean space can be described by its Gauss or mean curvatures. The curvedness of a Riemann space is much harder to describe since there is usually no or need a high dimensional surrounding space from which one can observe this curvedness from outside. Fortunately, the Riemann curvature tensor which characterizes the curvedness of the space can be calculated from the metric tensor G [6]:

$$\begin{aligned} R_{ijkl} &= \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j\partial x^k}+\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}-\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}-\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}\right) \\ &\quad +\sum_{m,n} g_{mn}\left(\Gamma^m_{jk}\Gamma^n_{il}-\Gamma^m_{jl}\Gamma^n_{ik}\right) \end{aligned} \quad (3)$$

In 2D spaces, due to the symmetry of the Riemann curvature tensor, there is only one nonzero and independent entry

$$\begin{aligned} R_{1212} &= \frac{1}{2}\left(\frac{\partial^2 g_{12}}{\partial x^2\partial x^1}+\frac{\partial^2 g_{21}}{\partial x^1\partial x^2}-\frac{\partial^2 g_{22}}{\partial x^1\partial x^1}-\frac{\partial^2 g_{11}}{\partial x^2\partial x^2}\right) \\ &\quad +\sum_{m,n=1}^{2} g_{mn}\,(\Gamma^m_{21}\Gamma^n_{12}-\Gamma^m_{22}\Gamma^n_{11}) \end{aligned} \quad (4)$$

which is equal to the Gaussian curvature of the embedding surface of the 2D Riemann space into a 3D Euclidean space.

According to the comparison theory of Riemann manifolds e.g. [3], one knows that if the curvature of a 2D Riemann manifold is negative everywhere, then the geodesics will never intersect each other or the exponential map is a diffeomorphism. Such a Riemann manifold is called a Cartan-Hadamard manifold. This means that for the whole color space with negative or zero curvature everywhere, there is a global Riemann normal coordinate system which can be regarded as an extended Munsell system or a generalization of the polar coordinate system in a Euclidean space. This theory extends also to higher dimensional space but for simplicity we only consider the 2D case hereafter.

However, if the curvature is positive then the geodesics will intersect somewhere, and therefore the exponential map will not be an homeomorphism. Hence there does not exist a single global Riemann normal coordinates for the whole color space.

Curvature in CIELUV space

It is known that the MacAdam's ellipses on the CIE xy chromaticity diagram have both positive and negative curvature [14].

Here we consider the curvature on the $L=50$ plane in the CIELUV space. We first investigate the geometry defined by the 25 ellipses measured by MacAdam. We interpolated the long axes, the short axes and the angles of the just-noticeable-difference (jnd) ellipses in the CIELUV space using the Akima algorithm [1]. The result is shown for the convex hull of the data points in Fig. 1. It can be observed that the interpolated ellipses have a quite uniform distribution. The Gaussian curvature of Fig.1 is shown in Fig.2. Here curvatures are calculated according to eq.(4), where the 1st and 2nd order differentiations are obtained by convolution of the metric tensor with a Gaussian kernel with standard deviation one subjected to the same 1st and 2nd order differentiation. In all figures below, we illustrate a negative value of curvature with blue color, and positive value of curvature with red color, the absolute values of the curvatures are proportional to the scale of each color. One can observe that the curvature is negative inside the convex hull of the jnd ellipses. The Riemann normal coordinates can therefore be constructed using geodesics starting in D65 the CIELUV space as shown in Fig.3(See also [5]).

Curvature in CIELAB space

Next we repeat the calculations but now use CIELAB coordinates. First, the 25 MacAdam ellipses are interpolated as in Fig.4 in the same way as in the CIELUV space. We observed however that the interpolated ellipses are not as uniformly distributed as in the CIELUV space. The curvature of the chromaticity plane in the CIELAB space is calculated as in Fig.5. It

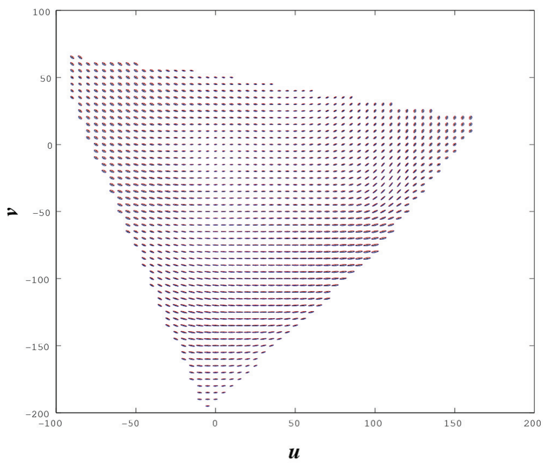

Figure 1. *Interpolation of MacAdam's ellipses in CIELUV*

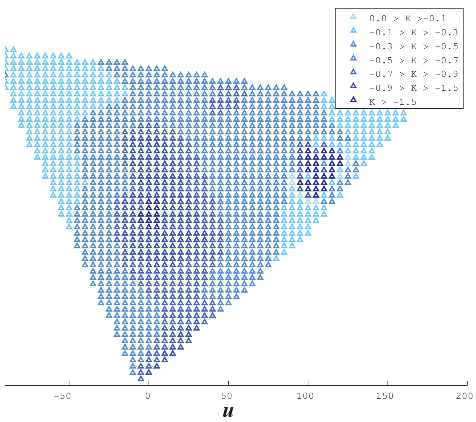

Figure 2. *Curvature of MacAdamn ellipses in CIELUV space*

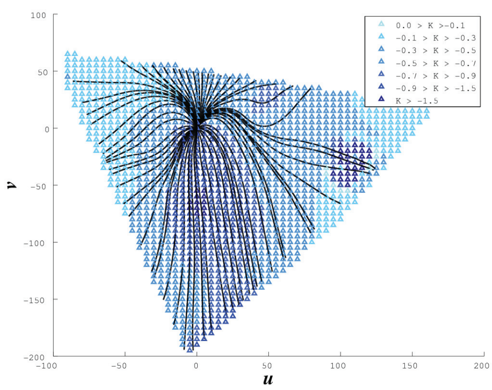

Figure 3. *Riemann normal coordinates in CIELUV space using MacAdam's ellipses*

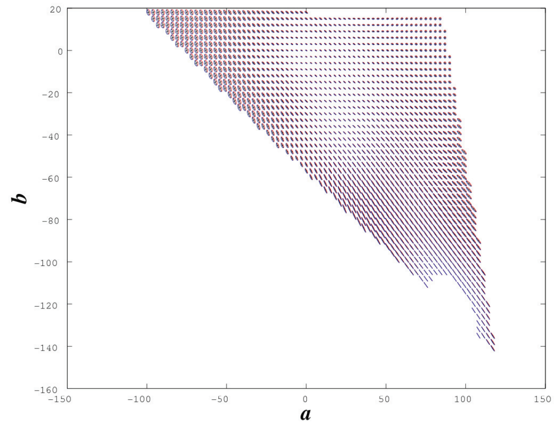

Figure 4. *Interpolation of MacAdam ellipses in CIELAB space*

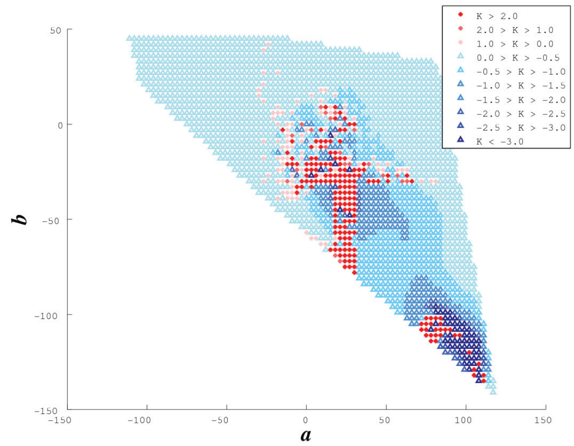

Figure 5. *Curvature of MacAdamn ellipses in CIELAB space*

can be observed that there are areas with large positive curvature values. Therefore, if one tries to draw geodesics using the metric in Fig.5, these geodesics intersect around the regions with positive curvature as can be seen in Fig.6. This illustrates that in the CIELAB case one cannot, in general, construct Riemann normal coordinates within the convex hull of the ellipses.

We also used another set of jnd ellipses in the CIELAB space by G. Cui et. al. (see [8]). The results obtained based on these measurements are shown in Fig.7. These ellipses are also interpolated inside the convex hull of the data points as in Fig.8. Certain non-uniformness can also be observed in the ellipses distribution. The curvature calculated from the metric obtained from Fig.8 is shown in the Fig.9. Once again, one can observe that there are several red areas indicating regions of positive curvatures. Again, if one tries to draw geodesics in order to build the Riemann normal coordiantes, intersections between geodesics occur around the areas with positive curvature, as shown in Fig.10.

Also here, this prevents the construction of the Riemann normal coordinates for the complete region under investigation.

Smoothing in CIELAB space

One possible way try to avoid the problem with the positiveness of the curvature is to smooth the metric or ellipses interpolation. We smoothed the metric data by G. Cui et. al. using a Gaussian filter of standard deviation 12 with a neighborhood of

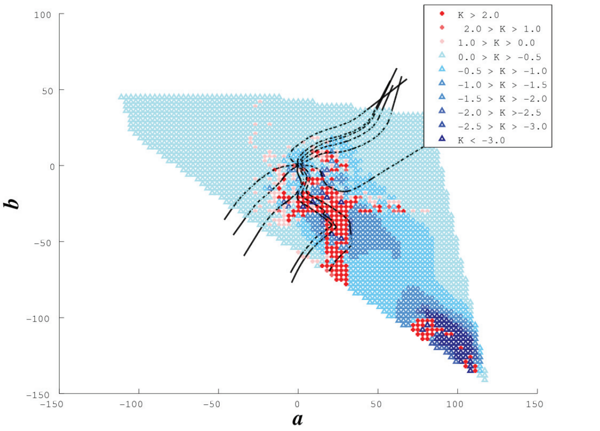

Figure 6. *Geodesics intersect each others in CIELAB space around positive curvature areas*

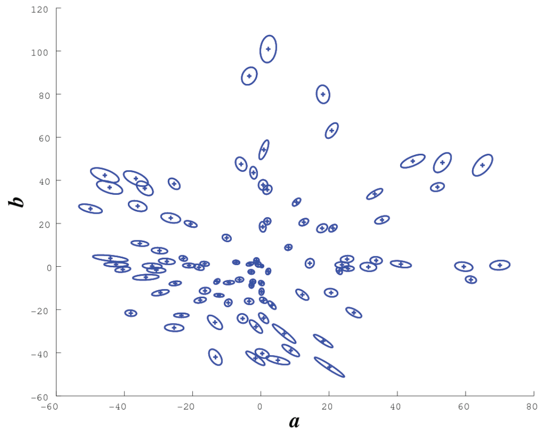

Figure 7. *The threshold ellipses by G. Cui et. al. in CIELAB space*

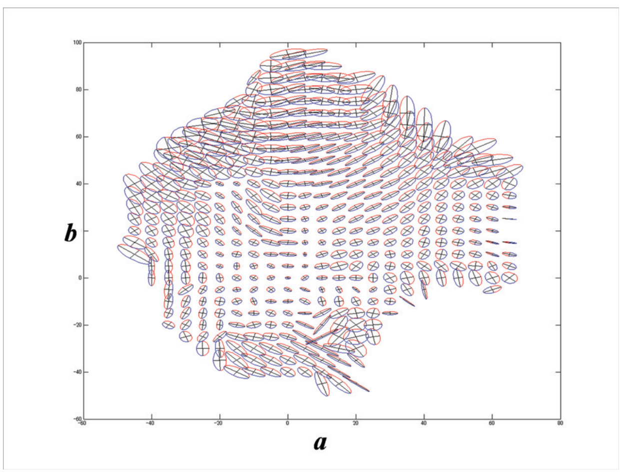

Figure 8. *Interpolation of ellipses by G. Cui et. al. in CIELAB space*

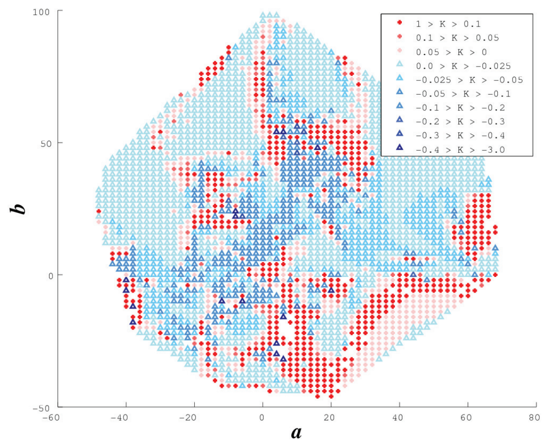

Figure 9. *Curvature of the ellipses by G. Cui et. al. in CIELAB*

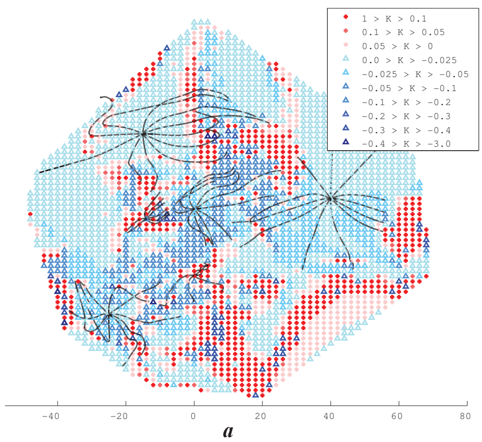

Figure 10. *Intersection of geodesics in CIELAB using ellipses by G. Cui et. al.*

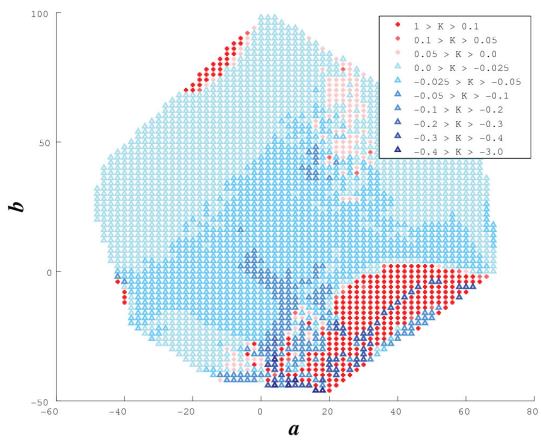

Figure 11. *Curvature of smoothed ellipses by G. Cui et. al. in CIELAB*

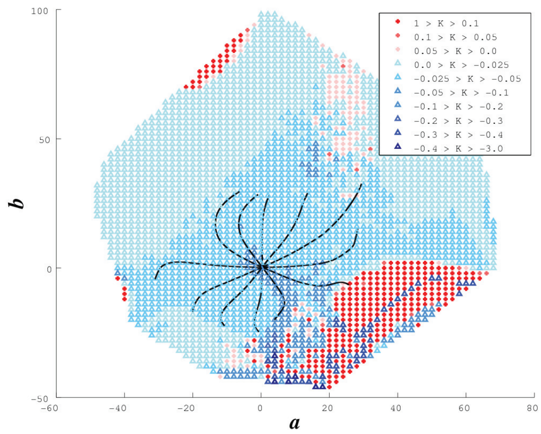

Figure 12. *Riemann normal coordinates in area of negative curvature after smoothing*

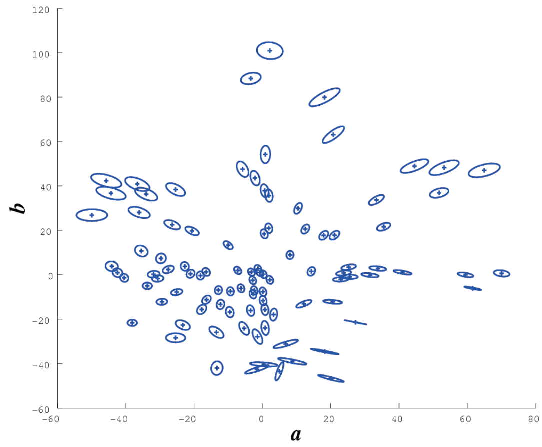

Figure 13. *Jnd threshold ellipses after smoothing*

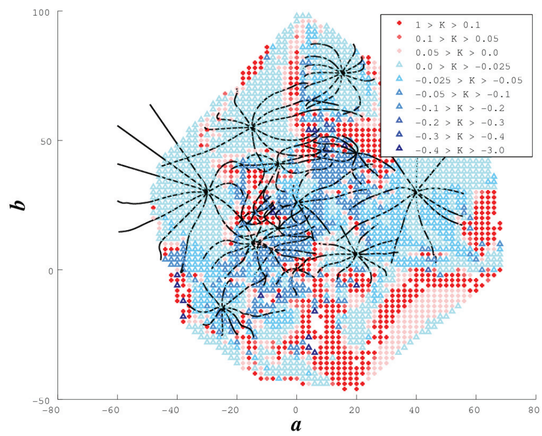

Figure 14. *A multi-patch Riemann normal coordinate systems in CIELAB space for ellipses by G. Cui et.al.*

size 97×97. The curvature after smoothing is shown in Fig.11. After smoothing the small positive curvature areas of the central region are removed. Thus one can draw geodesics, and therefore build a Riemann normal coordinate system inside this region of negative curvature as shown in the Fig.12. The down-right part in Fig.11 with positive curvature is actually the area in which the data points are very sparse. In general smoothing may reduce fluctuations of curvature. In particular, the absolute values of curvatures are reduced or both positive and negative curvatures become near zero. On the other hand, large scale features in total distribution of curvature remained.

Meanwhile, the sizes of the threshold ellipses become smaller after smoothing as shown in Fig.13 where they are five times magnified. The shapes of jnd ellipses change also considerably after the smoothing operation, which could be undesirable in certain cases.

Riemann normal coordinates in spaces with positive curvature

Assume that a color space contains parts with positive curvature. Then, according to the above investigation, it is impossible to construct a single system of Riemann normal coordinates for the whole space. In the following we present an algorithm that can be used to build a multi-patch coordinate system in color spaces that contain areas of positive curvature.

For a 2D Riemann space M, denote by ρ the length of geodesics from the same starting point to their intersection point. According to Rauch's theorem [3], let K be the curvatures of M, if $0 < L \leq K \leq H$, then

$$\frac{\pi}{\sqrt{H}} \leq \rho \leq \frac{\pi}{\sqrt{L}}. \tag{5}$$

In order to build a coordinate system for the whole space, we will use more than one Riemann normal coordinate system. The calculations can be done in parallel allowing fast implementations for practical applications and also reduce numerical integral errors.

In this construction, one first calculates the curvature and then estimates the minimal length ρ_{min}, which gives us the distance between the origins of the coordinate systems. One can determine the positions of the origins so that they are separated within twice of the ρ_{min}. All geodesics will be stopped less than ρ_{min}. Then one can apply a coordinates transformation between the different Riemann normal coordinates so that a global coordinates of an arbitrary point can be obtained.

Figure 14 shows a multi-patch Riemann normal coordinates in CIELAB space using ellipses by G. Cui et. al.. In fact, the smoothing can be combined with the multi-patch strategy. Since smoothing reduced the value of the positive curvature, the size of each patch is enlarged and the number of the patches is reduced.

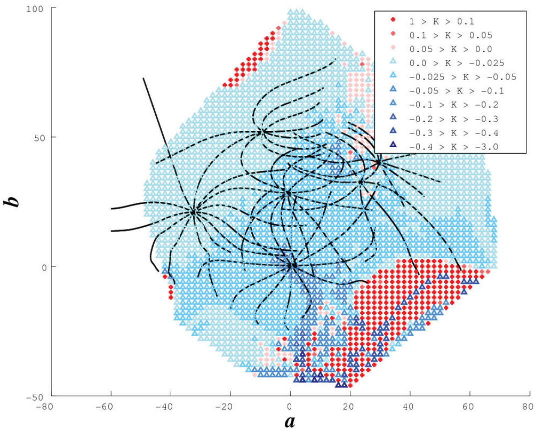

Figure 15. *A multi-patch Riemann normal coordinate systems in CIELAB after smoothing ellipses by G. Cui et. al.*

A trade off is then possible between acuracy after smoothing and efficiency of multi-patch coordinates construction. A multi-patch Riemann normal coordinate system is shown in Fig.15.

Summary and Conclusions

We have shown that the curvature of a color space measures a very important geometric property of the space. We also discussed why this has very important theoretical and practical implications for color science and technology. In particular, we have shown that a Munsell-like coordinates or Riemann normal coordinates only exist for a color space with negative or zero curvature everywhere.

We used two sets of jnd threshold ellipses and investigated properties of the color spaces constructed from them. We showed that there are significant differences between the spaces based on the CIELUV and the CIELAB system, and both of them are quite different from a uniform color space in the sense that instead of having zero curvature, they have either positive and negative curvatures. Our experiments with interpolation methods also showed that interpolation between discrete measurement points is an important factor. One alternative to use another metric in the space of symmetric positive-definite matrices in the construction ([7]). This construction is based on the observation that ellipses are the solutions of equations $x'Cx = 1$ where x is the coordinate vector and C is a symmetric positive-definite matrix. In [7] it is shown that it is possible to find a metric in the space of such matrices C such that the distance $d(C_1, C_2)$ between two matrices C_1, C_2 is invariant under all linear coordinate transformations, i.e. $d(C_1, C_2) = d(T'C_1T, T'C_2T)$ for all non-singular matrices T. For this metric it is possible to give an algorithmic description of the construction of geodesics in this space. Using this metric in the space of symmetric positive-definite matrices it is possible to introduce alternative interpolation strategies that have a geometrical background in the properties of the matrix space.

We also presented a strategy to build a global coordinates using multi-patch geodesics for color spaces with regions of positive curvature. Future research will include the investigation of curvature computed from other measurement data and models and we will also investigate the influence of different interpolation schemes on the final results.

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