Object-colour space revisited

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Abstract

Colorimetry can predict which lights will look alike. Such lights are called metameric. Two lights are known to be metameric if they have the same tri-stimulus values. Using the tri-stimulus values as the Cartesian coordinates one can represent light colours as points in a 3D space (referred to as the colorimetric space). All the light colours make a tri-dimensional manifold which can be represented as a circular cone in the colorimetric space. Furthermore, colorimetry can also predict which reflecting objects illuminated by the same light will look alike: those which reflect metameric lights. All the object colours can be represented as a closed solid inscribed in the light colour cone provided the illumination is fixed. However, when there are multiple illuminants the reflected light metamerism does not guarantee that the reflecting objects will look identical (referred to as the colour equivalence). In this paper three axioms are presented that allow the derivation of colour equivalence from metamerism. The colour of a reflecting object under various illuminations is shown to be specified by six numbers (referred to as its six-stimulus values). Using the six-stimulus values one can represent the colours of all the reflecting objects illuminated by various illuminants as a cone in the 6D space over the 5D ball.

Introduction

Colour spaces currently in use (e.g., CIE 1931, sRGB, CIELAB) are based on the colorimetric space which is suitable for representing the colour of light, and less so the colour of objects. The colorimetric space is the results of identifying ("gluing together") metameric (i.e., visually indistinguishable) lights. Formally, denote $\mathscr{P}$ the set of lights. It should be said that depending on the issue at hand I will consider various light sets. Generally, $\mathscr{P}$ will be the set of spectral power distributions, (i.e., integrable functions $p(\lambda)$ of wavelength). Let $\approx$ be metamerism (i.e., the equivalence relation on $\mathscr{P}$ satisfying Grassmann's laws [1]). Then, the quotient set $\mathscr{P}/\approx$ (i.e., the set of the classes of metameric lights) will be the set of light colours. It has been proved that there exist three functions $s_1(\lambda), s_2(\lambda)$, and $s_3(\lambda)$, which determine for each light $p(\lambda)$ three numbers, $\varphi_1(p), \varphi_2(p)$, and $\varphi_3(p)$:

$$\varphi_i(p) = \int_{\lambda_{\min}}^{\lambda_{\max}} p(\lambda) s_i(\lambda) d\lambda, \quad (1)$$

where $[\lambda_{\min}, \lambda_{\max}]$ is the visible spectrum interval; such that two lights $p_1(\lambda)$ and $p_2(\lambda)$ are metameric if and only if $\varphi_i(p_1) = \varphi_i(p_2)$ for each $i = 1,2,3$ [1]. I consider a particular case when $s_1(\lambda), s_2(\lambda)$, and $s_3(\lambda)$ are the CIE 1931colour matching functions [2], thus, the $\varphi_1(p), \varphi_2(p)$, and $\varphi_3(p)$ are the XYZ tri-stimulus values for $p(\lambda)$.

It should be noted that Eqs 1 makes sense also for the Dirac delta-function $\delta(\lambda - \lambda_0)$ the physical meaning of which is the monochromatic light of wavelength λ_0. So, sometimes $\mathscr{P}$ will be considered as a broader set including, if necessary, monochromatic lights too.

Using the XYZ tri-stimulus values as the Cartesian coordinates one can represent the colour of a light as a point in the 3D space (referred to as the *XYZ colorimetric space*). When $p(\lambda)$ runs through all the lights the triplets $(X(p), Y(p), Z(p))$ make a *colour cone* [1][2]. Therefore, the set of light colours can be represented as a cone in the colorimetric space.

When there is only one light source illuminating a scene, reflecting objects are visually indistinguishable (i.e., they have the same colour) if and only if they reflect metameric lights (i.e., they bring about the same XYZ tri-stimulus values) [2]. Therefore, there is a one-to-one correspondence between the set of object-colours in a single-illuminant scene and the set of the XYZ tri-stimulus values produced by all the possible reflecting objects. The latter is represented in the XYZ colorimetric space as a closed convex volume (referred to as the *object-colour solid*) inscribed in the colour cone [3].

In a multiple-illuminant scene, the metamerism of reflected lights does not guarantee equality of colour appearance of reflecting objects. For example, a yellow surface under a blue light and a blue surface under a yellow light might reflect lights metameric to the light reflected by a grey surface under day-light, but nevertheless, all the three surfaces might look very different from each other [4][5]. Abundant demonstrations of this sort testify that reflecting objects producing equal XYZ tri-stimulus values could have very different colour appearances [6][7]. Thus, it is worth distinguishing between metamerism and *colour equivalence*, that is, total apparent identity of the colour appearance of reflecting objects. The operational criterion for colour equivalence is similar to that for metamerism. Consider a bipartite field, the spectral reflectance properties of the two halves of which are different. Assume that each half is independently illuminated by lights with different spectral power distributions. Then colour equivalence of the two halves will mean the observer is unable to see the border between the halves. (Note that the bipartite field is not supposed to be displayed in isolation. On the contrary, it is assumed to be installed in a real scene alongside other objects).

When illumination is constant, metamerism and colour equivalence are the same, but for reflecting objects under different lights, metamerism does not necessarily imply colour equivalence. In other words, metamerism is a necessary but not sufficient condition for colour equivalence. Thus, being an efficient tools for specifying colour of light, the classical colorimetry looses its explanatory power when it comes to the colour of objects in multiple-illuminant scenes.

The main objective of this report is to extend the principles of colorimetry to make it applicable to multiple-illuminant scenes. This means, firstly, to formally define colour equivalence alongside metamerism; secondly, to establish some formal rules allowing us to predict which reflecting objects illuminated by which lights will be 'colour equivalent'; and thirdly, to develop a geometrical representation of the object-colours suitable for multiple-illuminant variegated scenes.

Colour equivalence

In what follows only matt Lambertian spatially homogeneous reflecting objects will be taken into consideration. The spectral power distribution of the light reflected by such objects can be expressed as the object spectral reflectance function (denoted $x(\lambda)$) times the spectral power distribution of the incident light (denoted $p(\lambda)$). Let $\mathscr{X}$ be a set of spectral reflectance functions, $x(\lambda)$, and $\mathscr{P}$ a set of spectral power distributions, $p(\lambda)$. For the reasons explained elsewhere [8], $\mathscr{P}$ will be restricted to the set of positive spectral power distributions, (i.e., such functions $p(\lambda)$ of wavelength that $p(\lambda) > 0$ for each λ). For brevity I will refer to $\mathscr{X}$ as the *object set*, $\mathscr{P}$ as the *light set*, and their elements as *objects* and *lights* respectively. Shortened notations x and p will be used for elements of $\mathscr{X}$ and $\mathscr{P}$, respectively. The Cartesian product $\mathscr{X} \times \mathscr{P}$, that is, the set of "object/light" pairs (x, p) (referred to as *colour stimuli*), will be referred to as the *colour stimulus set*.

Let us denote $\approx$ metamerism, and $\sim$ colour equivalence. More specifically, $\approx$ is an equivalence relation on $\mathscr{P}$ satisfying Grassmann's laws [1], and $\sim$ an equivalence relation on $\mathscr{X} \times \mathscr{P}$. That is, given a pair of objects x_1, and x_2 illuminated by lights p_1, and p_2, $(x_1, p_1) \sim (x_2, p_2)$ designates that these objects have the same colour appearance. In other words, they are completely visually indistinguishable. The quotient space $(\mathscr{X} \times \mathscr{P}) / \sim$ (i.e., the set of the classes of colour equivalent "object/light" pairs) will be referred to as the *object-colour set*.

As mentioned above, there is every indication that for single illuminant scenes colour equivalence coincides with metamerism; that is, speaking formally, the following axiom holds true.

Axiom 1 *For any particular light $p \in \mathscr{P}$, and every pair of objects x_1, and x_2 in $\mathscr{X}$*

$$(x_1, p) \sim (x_2, p) \Leftrightarrow x_1 p \approx x_2 p, \tag{2}$$

where $x_1 p$ and $x_2 p$ stand for the lights reflected from the objects x_1 and x_2.

Likewise, it seems safe to assume that two fragments of the same reflecting object illuminated by two lights will look identical in colour if they reflect metameric lights.

Axiom 2 *For any particular object $x \in \mathscr{X}$, and every pair of lights p_1, and p_2 in $\mathscr{P}$*

$$(x, p_1) \sim (x, p_2) \Leftrightarrow x p_1 \approx x p_2, \tag{3}$$

where xp_1 and xp_2 stand for the lights reflected from the object x.

Note that colour equivalence cannot be decomposed into two equivalence relations: one defined on the object set $\mathscr{X}$, and another on the light set $\mathscr{P}$. More specifically, assume that there exists (i) an equivalence relation $\sim_x$ on $\mathscr{X}$; and (ii) an equivalence relation $\sim_p$ on $\mathscr{P}$ such that for any $x_1, x_2 \in \mathscr{X}$, and $p_1, p_2 \in \mathscr{P}$

$$(x_1, p_1) \sim (x_2, p_2) \Leftrightarrow (x_1 \sim_x x_2) \text{ and } (p_1 \sim_p p_2). \tag{4}$$

In this case if for some light $p' \in \mathscr{P}$ the colour stimuli (x_1, p') and (x_2, p') are colour equivalent (i.e., $(x_1, p') \sim (x_2, p')$), then for any other light $p \in \mathscr{P}$ the colour stimuli (x_1, p) and (x_2, p) should be colour equivalent too. Thus, the reflected lights $x_1 p$ and $x_2 p$ must be metameric for any light p. However, this is impossible because, as well established, two objects reflecting metameric lights under one illuminant can reflect non-metameric lights under the other. This phenomenon is known as *metamer mismatching* [2]. It follows that colour equivalence cannot be reduced to an object-colour equivalence and a light colour equivalence which are independent of each other. In other words, colour equivalence is not separable.

An important implication of the colour equivalence inseparability is that it undermines the notion of the intrinsic colour of the surface, which suggests that colour is a property of surfaces [9][10]. If Eq. 4 held true the notion of the intrinsic colour would be justified. However, the colour equivalence inseparability unequivocally testifies that colour is a property of object/light pairs, not reflecting objects themselves.

While colour equivalence $\sim$ cannot be reduced to metamerism $\approx$, Axioms 1 and 2 together with a simple principle (which will be referred below as the impossibility of asymmetric colour matching) allow the derivation of colour equivalence $\sim$ from metamerism $\approx$. In this derivation the notion of a colour atlas [8] plays an important role.

A subset $\mathscr{A}_x$ in the object set $\mathscr{X}$ will be called an *object colour atlas* if for any light $p \in \mathscr{P}$, firstly, for every object x in $\mathscr{X}$ there is a single object, a_x, in $\mathscr{A}_x$ such that $(x, p) \sim (a_x, p)$, and secondly, all the objects in $\mathscr{A}_x$ are not colour equivalent, that is, for any $a_1, a_2 \in \mathscr{A}_x$ $a_1 p \not\approx a_2 p$.

Note that the definition of an object colour atlas can be reformulated in terms of metamerism because the colour equivalence is used in this definition only with respect to a single illuminant. Specifically, $\mathscr{A}_x$ is an object colour atlas if for any object $x(\lambda)$ and light $p(\lambda)$, firstly, there is a unique $a_x(\lambda)$, in $\mathscr{A}_x$ such that the lights $x(\lambda)p(\lambda)$ and $a_x(\lambda)p(\lambda)$ metameric; and secondly, if under any illumination different elements of $\mathscr{A}_x$ reflect non-metameric lights.

As pointed out elsewhere [8], an object colour atlas must include all the so-called optimal spectral reflectance functions, that is, those that map to the object-colour solid boundary. Indeed, an optimal spectral reflectance function cannot be metameric to any other reflectance function for any positive illuminant [8]. Therefore, for an optimal spectral reflectance function x the relation $(x, p) \sim (a_x, p)$ holds true only if $x = a_x$.

An important example of an object colour atlas is the so-called *optimal object colour atlas* [8], that is, a set of spectral reflectance functions of the form of:

$$(1 - \alpha) x_{0.5}(\lambda) + \alpha x_{opt}(\lambda), \tag{5}$$

where $0 \leq \alpha \leq 1$; $x_{opt}(\lambda)$ is an optimal spectral reflectance function; and $x_{0.5}(\lambda)$ is the spectral reflectance function taking 0.5 at every wavelength λ within the visible spectrum interval.

The notion similar to object colour atlas can be formally defined for lights as well. A subset of spectral power distributions $\mathscr{A}_p \subset \mathscr{P}$ will be called a *light colour atlas* if for any object $x \in \mathscr{X}$, and every light p in $\mathscr{P}$, firstly, there is a single element, a_p, in $\mathscr{A}_p$ such that $(x, p) \sim (x, a_p)$, and secondly, if all the elements in $\mathscr{A}_p$ are not colour equivalent for any object in $\mathscr{X}$, that is, $x a_1 \not\approx x a_2$ for any $a_1, a_2 \in \mathscr{A}_p$.

As established in colorimetry, each light is metameric to a mixture of a neutral light with either a monochromatic light or a light in the so-called "purple interval", that is, the mixtures of the two monochromatic lights at the ends of the visible spectrum [2]. Formally, given a neutral light $p_n(\lambda)$, each light is metameric to the following mixture

$$q_n p_n(\lambda) + q p(\lambda), \tag{6}$$

where $p(\lambda)$ is either a monochromatic light $\delta(\lambda - \mu)$ of some wavelength μ, or a mixture of the two monochromatic lights:

$\delta(\lambda-\lambda_{\max})$ and $\delta(\lambda-\lambda_{\min})$; q_0 and q are non-negative numbers.

At first glance, a subset of lights (6) is an obvious candidate to be a light colour atlas. Nevertheless, it does not make a light colour atlas because two optimal spectral reflectance functions become metameric under a monochromatic light of the wavelength λ if these reflectance functions take zero at λ. (This is one of the reasons why the monochromatic lights are excluded from $\mathscr{P}$ in the present context.)

Still, a monochromatic light $\delta(\lambda-\mu)$ can be approximated by a light with Gaussian spectral power distribution of the form of

$$\left(\sqrt{2\pi}\sigma\right)^{-1}\exp\left(-((\lambda-\mu)/\sigma)^2\right), \quad (7)$$

letting σ approach zero. Therefore, replacing monochromatic lights in Eq. 6 with Gaussians (7) with sufficiently small σ one can get a proper approximation to the light colour atlas. Admittedly, such a set of lights will be a light colour atlas for only a subset of $\mathscr{P}$. This subset does not include lights of very narrow-band spectral power distributions. Yet, for sufficiently small σ the difference between this subset of $\mathscr{P}$ and the whole $\mathscr{P}$ can be made negligibly small. I will refer to the set of lights of the form (6) where monochromatic lights are replaced with Gaussians (7) as the *pseudo-monochromatic light colour atlas*.

Consider an object colour atlas $\mathscr{A}_x$ and a light colour atlas $\mathscr{A}_p$. For any object $x\in\mathscr{X}$ illuminated by any light $p\in\mathscr{P}$ there exists an element a_x in $\mathscr{A}_x$, and an element a_p in $\mathscr{A}_p$ such that the pairs (x,p) and (a_x,a_p) are colour equivalent, that is, $(x,p)\sim(a_x,a_p)$. Indeed, there exists an $a_x\in\mathscr{A}_x$ such that $(x,p)\sim(a_x,p)$, and there exists an $a_p\in\mathscr{A}_p$ such that $(a_x,p)\sim(a_x,a_p)$. Thus, we have $(x,p)\sim(a_x,a_p)$.

If there are no colour equivalent pairs in $\mathscr{A}_x\times\mathscr{A}_p$ then for any object $x\in\mathscr{X}$ illuminated by any light $p\in\mathscr{P}$ there is exactly one pair $(a_x,a_p)\in\mathscr{A}_x\times\mathscr{A}_p$ such that $(x,p)\sim(a_x,a_p)$ because of the transitivity of the colour equivalence. In this case the Cartesian product of the object and light atlases $\mathscr{A}_x\times\mathscr{A}_p$ (referred to as just the *colour atlas*) uniquely represents all the classes of colour equivalence. Thus, the colour atlas $\mathscr{A}_x\times\mathscr{A}_p$ can be considered as a representative of the object-colour set $(\mathscr{X}\times\mathscr{P})/\sim$. I will call elements of the colour atlas $\mathscr{A}_x\times\mathscr{A}_p$ *colour equivalent stimuli*. Each colour equivalent stimulus is a pair (a_x,a_p) where a_x belongs to the object colour atlas, and a_p to the light colour atlas. I will refer to a_x as the *material component*, and a_p as the *lighting component* of the colour equivalent stimulus (a_x,a_p).

The impossibility of colour equivalence in the colour atlas $\mathscr{A}_x\times\mathscr{A}_p$ means that asymmetric colour matching is impossible for the elements of the colour atlas. Indeed, if colour equivalence in the colour atlas $\mathscr{A}_x\times\mathscr{A}_p$ is impossible, then, given $a_x,a'_x\in\mathscr{A}_x$, and $a_p,a'_p\in\mathscr{A}_p$, $(a_x,a_p)\sim(a'_x,a'_p)$ implies $a_x=a'_x$, and $a_p=a'_p$. Consider colour stimuli (a_x,a_p) and (a_x,a'_p). If $a_p\neq a'_p$ then $(a_x,a_p)\nsim(a_x,a'_p)$. An asymmetric colour match means that there exists an a'_x such that $(a_x,a_p)\sim(a'_x,a'_p)$. That is, the colour difference induced by a difference in illumination can be compensated for by a difference in reflectance. If $(a_x,a_p)\sim(a'_x,a'_p)$ implies $a_x=a'_x$, and $a_p=a'_p$, then asymmetric colour matching is impossible in $\mathscr{A}_x\times\mathscr{A}_p$. Also there might be asymmetric colour matching of another kind when a colour difference induced by a difference in reflectance can be compensated for by a difference in illumination. Specifically, given $(a_x,a_p)\nsim(a'_x,a_p)$, there might exist a'_p such that $(a_x,a_p)\sim(a'_x,a'_p)$. However, the impossibility of colour equivalence in $\mathscr{A}_x\times\mathscr{A}_p$ excludes the existence of such a'_p.

That the colour difference induced by a difference in illumination cannot be compensated for by a difference in reflectance is a well-known fact established in colour constancy experiments [11][12]. For instance, Brainard, Brunt & Speigle ([11], p. 2098) described this as follows. "At this match point, however, the test and the match surfaces looked different, and the observers felt as if further adjustments of the match surface should produce a better correspondence. Yet turning any of the knobs or combinations of knobs only increased the perceptual difference."

Multidimensional scaling of coloured papers lit by lights of different chromaticity showed that the dissimilarity between such papers never became zero, even when the papers reflected metameric lights [4][5]. Furthermore, these multidimensional studies revealed more than three colour dimensions. Specifically, it was shown that there existed three material and three lighting dimensions of object colour [4][5][12]. This is in line with early results indicating that human observers can distinguish between a colour difference produced by a material difference as compared to that produced by a difference in lighting [13]. Thus, manipulating material properties such as reflectance can only minimise the colour difference in the material colour dimensions, but it cannot eliminate the colour difference in the lighting colour dimensions. Likewise, one cannot eliminate a colour difference induced by a difference in reflectance by manipulating the illumination. So, it is safe to assume that the impossibility of asymmetric colour match is an important feature of human colour vision, that will be formalised as the following axiom.

Axiom 3 *Given an object colour atlas $\mathscr{A}_x$ and a light colour atlas $\mathscr{A}_p$, we will say that* impossibility of asymmetric colour match *takes place if the following property holds true for any $a_x,a'_x\in\mathscr{A}_x$, and $a_p,a'_p\in\mathscr{A}_p$:*

$$\left((a_x,a_p)\sim(a'_x,a'_p)\right)\Rightarrow\left(a_p=a'_p\right)\ \textit{and}\ \left(a_x=a'_x\right),$$

where $\sim$ is the colour equivalence relation.

Axiom 3 secures that for every object with the spectral reflectance function $x(\lambda)$ illuminated by a light with the spectral power distribution function $p(\lambda)$ there is a unique element in the colour atlas $\mathscr{A}_x\times\mathscr{A}_p$ colour equivalent to the colour stimulus (x,p).

It follows from the definition of colour atlas that there is one-to-one map between any two colour atlases. If we know the colour equivalent stimulus for some colour stimulus in one colour atlas then, in principle, one can derive the colour equivalent stimulus for this colour stimulus in any other colour atlas.

The choice of colour atlas amounts to the choice of non-linear coordinate system for the object-colour set. A change of colour atlas, then, amounts to a change of coordinate system in the object-colour set. The object-colour set along with the family of colour atlases considered as the family of coordinates systems will be referred to as the *object-colour manifold*.

The colour stimuli (x_1,p) and (x_2,p) may happen to be represented by colour equivalent stimuli differing only in their material components in one colour atlas but not in the other. To be more exact, given two colour atlases $\mathscr{A}_x\times\mathscr{A}_p$ and $\mathscr{A}'_x\times\mathscr{A}'_p$, colour stimuli (x_1,p) and (x_2,p) can be represented, on the one hand, as $(x_1,p)\sim(a_{m1},a_l)$, and $(x_2,p)\sim(a_{m2},a_l)$, where $a_{m1},a_{m2}\in\mathscr{A}_x$; $a_l\in\mathscr{A}_p$; and on the other as $(x_1,p)\sim(a'_{m1},a'_{l1})$, and $(x_2,p)\sim(a'_{m2},a'_{l2})$, where $a'_{m1},a'_{m2}\in\mathscr{A}'_x$; $a'_{l1},a'_{l2}\in\mathscr{A}'_p$ such that $a'_{l1}\neq a'_{l2}$. Therefore, the equality of the material (respectively, lighting) components is not a property invariant with respect to the choice of colour atlas. In other words, it is not a property of colour, it is a property of the representation of colour by the colour atlas.

Interestingly, a human observer's ability to distinguish between material and lighting colour differences [4][5][12][13] leads to the conjecture that the human visual system might encode colour using a particular colour atlas. An intriguing issue is what this colour atlas is.

Evaluation of colour equivalent stimuli: Two-step colour matching

Consider a colour atlas $\mathscr{A}_x \times \mathscr{A}_p$ and a colour stimulus (x,p). There is a unique element (a_m,a_l) in $\mathscr{A}_x \times \mathscr{A}_p$ which is colour equivalent to this particular colour stimulus, that is,

$$(x,p) \sim (a_m,a_l). \tag{8}$$

It is worth mentioning that the material a_m and lighting component a_l in (8) can be evaluated independently. Indeed, (8) can be decomposed into the following two colour equivalences

$$(x,p) \sim (a_m,p), \tag{9}$$

and

$$(a_m,p) \sim (a_m,a_l). \tag{10}$$

The material component a_m is such that it matches object x under the same illumination p. The lighting component a_l is such that it matches light p when both fall on the same object (a_m). Therefore, evaluation of the colour equivalent stimulus (a_m,a_l) can be performed as a sequence of two colour matchings. As both the matchings are symmetrical, the evaluation can be done either experimentally or theoretically.

If the atlases $\mathscr{A}_x$ and $\mathscr{A}_p$ can be implemented as real reflecting objects (e.g., papers) and lights, the colour equivalent stimulus (a_m,a_l) for a given colour stimulus (x,p) can be established experimentally in two steps. First, a symmetrical colour matching experiment is carried out to determine one of the two components (either a_m or a_l); and then the second component is determined in a second symmetrical colour matching experiment.

When the atlases $\mathscr{A}_x$ and $\mathscr{A}_p$ are impossible to implement (as in the case of the optimal object colour atlas), the colour equivalent stimulus (a_m,a_l) for a given colour stimulus (x,p) can be evaluated theoretically. Indeed, as Eq. 9 means colour equivalence of the two objects (x and a_m) under the same illumination (p) it implies metamerism $xp \approx a_m p$, that is,

$$\varphi_i(xp) = \varphi_i(a_m p) \qquad (i = 1,2,3), \tag{11}$$

where φ_i is defined by Eqs 1. Likewise, Eq. 10 implies metamerism $a_m p \approx a_m a_l$, that is,

$$\varphi_i(a_m p) = \varphi_i(a_m a_l) \qquad (i = 1,2,3). \tag{12}$$

Equations 11 and 12 implicitly determine a_m and a_l. Moreover, as both the material and light colour atlases are tri-dimensional sets, their elements can, in principle, be specified by three parameters (coordinates) each. This will reduce the evaluation of a colour equivalent stimulus (a_m,a_l) to resolving a system of six simultaneous equations with respect to six unknowns.

Consider, for example, the colour atlas $\mathscr{A}_x \times \mathscr{A}_p$, where $\mathscr{A}_x$ is the optimal object colour atlas (see Eq. 5), and $\mathscr{A}_p$ is the pseudo-monochromatic light colour atlas (see Eqs 6 and 7). In the case of the CIE 1931 colour matching functions the optimal spectral reflectance functions prove to be the rectangular spectral reflectance functions (i.e., taking only 0 or 1 with not more than two transitions between these values). As shown elsewhere [8], in this case the elements of the optimal object colour atlas have the form of:

$$x_{0.5}(\lambda) + \alpha\left(x(\lambda;\lambda_1,\lambda_2) - x_{0.5}(\lambda)\right) \tag{13}$$

where

$$x(\lambda;\lambda_1,\lambda_2) = \begin{cases} 1, & \text{if} \quad \lambda_1 \le \lambda \le \lambda_2; \\ 0, & \text{if} \quad \lambda < \lambda_1, \text{ or } \lambda > \lambda_2. \end{cases}$$

and $|\alpha| \le 1$ (referred to as the *rectangle colour atlas*).

The three numbers α, λ_1, and λ_2 uniquely determine an element of the rectangle colour atlas. Moreover, it has been found that Munsell Hue is well correlated with $\overline{\lambda} = 0.5(\lambda_1 + \lambda_2)$, and blackness/whiteness with $\delta = |\lambda_1 - \lambda_2|$ [8]. The parameter α corresponds to the subjective strength of the chromatic quality. The parameters α, δ, and $\overline{\lambda}$ will be referred to as *purity, spectral band*, and *central wavelength*, respectively. I will denote $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ the element of the rectangle colour atlas with purity α, spectral bandwidth δ, and central wavelength $\overline{\lambda}$.

Thus, when s_1, s_2, s_3 are the CIE 1931 colour matching functions equations (11) take the form of $(i = 1,2,3)$

$$\int_{\lambda_{\min}}^{\lambda_{\max}} x(\lambda)\,p(\lambda)\,s_i(\lambda)\,d\lambda = \int_{\lambda_{\min}}^{\lambda_{\max}} r\left(\lambda;\alpha,\delta,\overline{\lambda}\right) p(\lambda)\,s_i(\lambda)\,d\lambda, \tag{14}$$

and can be uniquely resolved with respect to α, δ, and $\overline{\lambda}$.

Having derived $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ from Eqs 14, it can be used to evaluate a_l, since evaluation of elements of the pseudo-momochromatic colour atlas can also be reduced to evaluating three parameters. Note that Eq. 6 can be interpreted as an algebraic linear combination (with the weights q_n, and q) of the neutral light $p_n(\lambda)$ and a monochromatic light of some wavelength λ_0. The weight q_n is always non-negative whereas the weight q can take negative values. Likewise, using Gaussian approximation (7), one can interpret Eq. 6 as an algebraic linear combination of the neutral light and a Gaussian with some wavelength μ. Although when $q < 0$, it defines no light, Eq. 6 can be used as a parametric representation of the pseudo-monochromatic colour atlas. The parameters q_n, q, and μ fully specify the elements of the pseudo-monochromatic light colour atlas. The denotation $g(\lambda;q_n,q,\mu)$ will be used for a function determined by Eq. 6 with $p(\lambda)$ defined by Eq. 7.

Replacing a_m and a_l in Eqs 12 with $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ derived from Eqs 14 and $g(\lambda;q_n,q,\mu)$ respectively, one gets the following equations $(i = 1,2,3)$:

$$\int_{\lambda_{\min}}^{\lambda_{\max}} r\left(\lambda;\alpha,\delta,\overline{\lambda}\right) p(\lambda)\,s_i(\lambda)\,d\lambda = \int_{\lambda_{\min}}^{\lambda_{\max}} r\left(\lambda;\alpha,\delta,\overline{\lambda}\right) g(\lambda;q_n,q,\mu)\,s_i(\lambda)\,d\lambda \tag{15}$$

which can be resolved with respect to q_n, q, and μ.

The six numbers $(\alpha,\delta,\overline{\lambda},q_n,q,\mu)$ specify object-colour in a similar to that with which the XYZ tri-stimulus values specify light colour in the classical colorimetry. Furthermore, as is the case for XYZ tri-stimulus values they can, in principle, be

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evaluated in a colour matching experiment. Indeed, a sextuplet $(\alpha,\delta,\overline{\lambda},q_n,q,\mu)$ specify a particular object illuminated by a particular light (i.e., a colour stimulus). Adjusting the parameters $(\alpha,\delta,\overline{\lambda},q_n,q,\mu)$ one can get, in principle, colour equivalence of this colour stimulus with any object lit by an arbitrary light. Although experiments of this kind have not been done as yet, there is nothing that could prevent us from conducting them. By analogy, I will refer to the sextuplet $(\alpha,\delta,\overline{\lambda},q_n,q,\mu)$ as the *six-stimulus values*, the triplet $(\alpha,\delta,\overline{\lambda})$ being referred to as the *material tri-stimulus* values, and (q_n,q,μ) the *lighting tri-stimulus values*.

Object-colour manifold

The rectangle colour atlas, $\mathscr{A}_x$, can be represented as a ball in the 3D Euclidean space [8]. For each of its elements $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ the central wavelength $\overline{\lambda}$ and spectral band δ specify the longitude and latitude of the element's location in the ball respectively, and the purity determines its distance from the ball centre [8]. In turn, the pseudo-monochromatic light colour atlas, $\mathscr{A}_p$, can be represented as an open 3D circular cone (i.e., the 3D circular cone without boundary). Hence, the colour atlas $\mathscr{A}_x \times \mathscr{A}_p$ can be represented as the Cartesian product of the 3D ball and the open 3D circular cone. It follows that the object-colour manifold can also be geometrically represented as the Cartesian product of the 3D ball and the open 3D circular cone.

Note that if $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ and $g(\lambda;q_n,q,\mu)$ satisfy equations (14) and (15) for some $x(\lambda)$ and $p(\lambda)$, then for any positive number k_0 the pair $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ and $g(\lambda;k_0q_n,k_0q,\mu)$ will satisfy equations (14) and (15) for $x(\lambda)$ and $k_0p(\lambda)$. In other words, changing only the intensity of the illumination results in only a corresponding multiplicative change in q_n, and q, the rest of the six-stimulus values remaining the same.

It follows that one can confines oneself to a "cross-section" of the cone of lights $\mathscr{P}$. For example, only equiluminant lights (i.e., lights of equal tri-stimulus value Y) can be taken into consideration. In this case the corresponding subset of the pseudo-monochromatic light colour atlas will be an open 2D set homeomorphic to (i.e., it can be continuously transformed into) an open 2D ball (i.e. an open disc). In other words, the pseudo-monochromatic light colour atlas for the equiluminant lights can be represented as an open disc. Therefore, in this case the object-colour manifold can be represented as the Cartesian product of the closed 3D ball and the open 2D ball. The "interior" of this product is homeomorphic to an open 5D ball. Thus, in the case of equiluminant lights the object-colour manifold can be represented as an open 5D ball.

Alternative parametric representations of the object-colour manifold

Although the set of functions $\{g(\lambda;q_n,q,\mu)\}$ determined by Eq. 6 with $p(\lambda)$ defined by Eq. 7, and q not always positive, is not strictly speaking a light colour atlas we found it useful for computational purposes as a parametric representation of the pseudo-momochromatic colour atlas. Depending on the context, some other parametric representations of the object colour atlas (thus, the object-colour manifold) can be used.

Bilinear representation

Given the colour atlas $\mathscr{A}_x \times \mathscr{A}_p = \left\{\left(r\left(\lambda;\alpha,\delta,\overline{\lambda}\right),g(\lambda;q_n,q,\mu)\right)\right\}$, consider three arbitrary spectral reflectance functions $x_1(\lambda), x_2(\lambda)$ and $x_3(\lambda)$, and three arbitrary spectral power distributions $p_1(\lambda), p_2(\lambda)$ and $p_3(\lambda)$.

Let us also consider the following equations ($i=1,2,3$):

$$\int_{\lambda_{\min}}^{\lambda_{\max}} r\left(\lambda;\alpha,\delta,\overline{\lambda}\right) g(\lambda;q_n,q,\mu)\, s_i(\lambda)\, d\lambda = \int_{\lambda_{\min}}^{\lambda_{\max}} \left(\sum_{j=1}^{3} k_j x_j(\lambda)\right) g(\lambda;q_n,q,\mu)\, s_i(\lambda)\, d\lambda, \quad (16)$$

and

$$\int_{\lambda_{\min}}^{\lambda_{\max}} \left(\sum_{j=1}^{3} k_j x_j(\lambda)\right) g(\lambda;q_n,q,\mu)\, s_i(\lambda)\, d\lambda = \int_{\lambda_{\min}}^{\lambda_{\max}} \left(\sum_{j=1}^{3} k_j x_j(\lambda)\right)\left(\sum_{m=1}^{3} k_{m+3} p_m(\lambda)\right) s_i(\lambda)\, d\lambda. \quad (17)$$

If these six equations can be uniquely resolved with respect to $k_1,\ldots,k_6$ for any sextuplet $(\alpha,\delta,\overline{\lambda},q_n,q,\mu)$, we will say that we have a *bilinear representation* of the atlas $\mathscr{A}_x \times \mathscr{A}_p$.

A sufficient condition for the existence of a unique solution for equations (16) and (17) is the non-singularity of the matrix $\{\omega_{ij}\}$ where

$$\omega_{ij} = \int_{\lambda_{\min}}^{\lambda_{\max}} x_i(\lambda)\, p(\lambda)\, s_j(\lambda)\, d\lambda \quad (18)$$

for any $p(\lambda)$.

The advantages of a bilinear representation have been highlighted by the authors of the so-called linear models [10][14]. However, an apparent disadvantage of these models is that they restrict the object and light sets to the three-dimensional subspaces spanned over three predetermined reflectance spectra and three lights. However, as one can see, such a restriction is unnecessary since the bilinear representation can be readily used for representing the whole (infinite-dimensional) colour stimulus set.

Gaussian representation

Given the colour atlas $\mathscr{A}_x \times \mathscr{A}_p = \left\{\left(r\left(\lambda;\alpha,\delta,\overline{\lambda}\right),g(\lambda;q_n,q,\mu)\right)\right\}$, consider a three-parameter set of spectral reflectance functions, $g_m(\lambda;k_m,\theta_m,\mu_m)$, which are defined as follows. Let us designate $\Lambda = \lambda_{\max} - \lambda_{\min}$. When $\mu_m \le (\lambda_{\max}+\lambda_{\min})/2$ we have: for $\lambda_{\min} \le \lambda \le \mu_m + \Lambda/2$

$$g_m(\lambda;k_m,\theta_m,\mu_m) = k_m \exp\left[-\theta_m(\lambda-\mu_m)^2\right]; \quad (19)$$

and for $\mu_m + \Lambda/2 < \lambda \le \lambda_{\max}$

$$g_m(\lambda;k_m,\theta_m,\mu_m) = k_m \exp\left[-\theta_m(\lambda-\mu_m-\Lambda)^2\right]. \quad (20)$$

When $\mu_m > (\lambda_{\max}+\lambda_{\min})/2$ we have: for $\lambda_{\min} \le \lambda \le \mu_m - \Lambda/2$

$$g_m(\lambda;k_m,\theta_m,\mu_m) = k_m \exp\left[-\theta_m(\lambda-\mu_m+\Lambda)^2\right] \quad (21)$$

and for $\mu_m - \Lambda/2 < \lambda \le \lambda_{\max}$

$$g_m(\lambda;k_m,\theta_m,\mu_m) = k_m \exp\left[-\theta_m(\lambda-\mu_m)^2\right] \quad (22)$$

For $0 \le k_m \le 1$, $\lambda_{\min} \le \mu \le \lambda_{\max}$, and positive θ_m equations (19 - 20) and (21 - 22) define a Gaussian on the visible spectrum circle with its maximum (equal to k_m) being at the wavelength μ.

Consider also a three-parameter set of spectral power distribution functions of the same form, that is, when $\mu_l \leq (\lambda_{\max}+\lambda_{\min})/2$

$$g_l(\lambda;k_l,\theta_l,\mu_l) = k_l \exp\left[-\theta_l(\lambda-\mu_l)^2\right] \quad (23)$$

if $\lambda_{\min} \leq \lambda \leq \mu_l + \Lambda/2$; and

$$g_l(\lambda;k_l,\theta_l,\mu_l) = k_l \exp\left[-\theta_l(\lambda-\mu_l-\Lambda)^2\right], \quad (24)$$

if $\mu_l+\Lambda/2 < \lambda \leq \lambda_{\max}$. Likewise, when $\mu_l > (\lambda_{\max}+\lambda_{\min})/2$

$$g_l(\lambda;k_l,\theta_l,\mu_l) = k_l \exp\left[-\theta_l(\lambda-\mu_l+\Lambda)^2\right] \quad (25)$$

if $\lambda_{\min} \leq \lambda \leq \mu_l - \Lambda/2$; and if $\mu_l - \Lambda/2 < \lambda \leq \lambda_{\max}$

$$g_l(\lambda;k_l,\theta_l,\mu_l) = k_l \exp\left[-\theta_l(\lambda-\mu_l)^2\right]. \quad (26)$$

Although there is no simple relationship between the parameters k_m, θ_m, and μ_m (respectively, k_l, θ_l, and μ_l), and the perceptual dimensions of the colours of objects (respectively, light), the peak wavelength μ_m (respectively, μ_l) can be considered, to a first approximation, as the stimulus correlate of chromatic hue.

If for any sextuplet $(\alpha,\delta,\overline{\lambda},q_n,q,\mu)$ the equations $(i=1,2,3)$

$$\int_{\lambda_{\min}}^{\lambda_{\max}} r\left(\lambda;\alpha,\delta,\overline{\lambda}\right) g(\lambda;q_n,q,\mu)\, s_i(\lambda)\, d\lambda = \int_{\lambda_{\min}}^{\lambda_{\max}} g_m(\lambda;k_m,\theta_m,\mu_m)\, g(\lambda;q_n,q,\mu)\, s_i(\lambda)\, d\lambda, \quad (27)$$

$$\int_{\lambda_{\min}}^{\lambda_{\max}} g_m(\lambda;k_m,\theta_m,\mu_m)\, g(\lambda;q_n,q,\mu)\, s_i(\lambda)\, d\lambda = \int_{\lambda_{\min}}^{\lambda_{\max}} g_m(\lambda;k_m,\theta_m,\mu_m)\, g_l(\lambda;k_l,\theta_l,\mu_l)\, s_i(\lambda)\, d\lambda, \quad (28)$$

have a unique solution with respect to the triplets (k_m,θ_m,μ_m) and (k_l,θ_l,μ_l), we will say that we have the *Gaussian representation* of the atlas $\mathscr{A}_x \times \mathscr{A}_p$.

The Gaussian spectral reflectance functions and spectral power distributions have been used before to model objects and lights respectively [15][16]. It turns out that they can be readily used to represent the whole colour stimulus set.

Trade-off between material and lighting components

If a pair of a rectangular spectral reflectance function $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ and a Gaussian $g(\lambda;q_n,q,\mu)$ make the colour equivalent stimulus for an object with the spectral reflectance function $x(\lambda)$ illuminated by a light with the spectral power distribution function $p(\lambda)$, the following equations hold true $(i=1,2,3)$:

$$\int_{\lambda_{\min}}^{\lambda_{\max}} r\left(\lambda;\alpha,\delta,\overline{\lambda}\right) g(\lambda;q_n,q,\mu)\, s_i(\lambda)\, d\lambda = \varphi_i(xp), \quad (29)$$

where $(\varphi_1(xp),\varphi_2(xp),\varphi_3(xp))$ are the XYZ tri-stimulus values (Eqs 1) produced by the light reflected from the object with the spectral reflectance function $x(\lambda)$ illuminated by the light with the spectral power distribution function $p(\lambda)$. Certainly, Eqs 29 do not determine uniquely the colour equivalent stimulus for the colour stimulus (x,p) because the number of unknown parameters exceeds the number of equations. It follows that if one assumes that the XYZ tri-stimulus values $(\varphi_1(xp),\varphi_2(xp),\varphi_3(xp))$ are the only source of information about the colour stimulus (x,p), its colour cannot be uniquely recovered from these XYZ tri-stimulus values.

In fact, for fixed XYZ tri-stimulus values $(\varphi_1(xp),\varphi_2(xp),\varphi_3(xp))$ Eqs 29 determines a three-dimensional manifold of colour equivalent stimuli (r,g) such that rg is metameric to xp. I will refer to it as the *material-lighting invariance manifold*.

The Gaussian parametric representation lends itself to looking into the material-lighting invariance manifold. Assume that $\mu_m,\mu_l \leq (\lambda_{\max}+\lambda_{\min})/2$, and $\lambda_{\min} \leq \lambda \leq \max(\mu_m,\mu_l) + (\lambda_{\max}-\lambda_{\min})/2$. In this case, for each i, equations analogous to (29) will take the form

$$\int_{\lambda_{\min}}^{\lambda_{\max}} k_m k_l \exp\left[-\theta_m(\lambda-\mu_m)^2\right] \exp\left[-\theta_l(\lambda-\mu_l)^2\right] \times s_i(\lambda)\, d\lambda = \varphi_i(xp). \quad (30)$$

It follows that any pair of k'_m and k'_l such that $0 \leq k'_m \leq 1$, $0 \leq k'_l$, and $k'_m k'_l = k_m k_l$ will satisfy Eqs 30. The coefficient k_l determines the brightness of the light with the Gaussian spectral power distribution (23). The coefficient k_m is a stimulus correlate of the lightness of the object with the Gaussian spectral reflectance function (19) – the smaller k_m, the blacker the object's colour. It has been suggested a long time ago that for a fixed light reaching the eye, first, there might be many combinations of lightness and brightness that can be perceived; and second, these perceptually coupled lightness and brightness reciprocally covary with each other (for a review see [17][18]). This hypothesis has been supported by experiments in which altering the apparent spatial layout has been shown to result in an apparent change in the object-colour. For example, rearranging depth cues one can make an object be illusorily perceived at a different distance where the illumination is different. It turns out that such an illusory depth shift results in a change in the object's colour. A white surface, after having been illusory moved to a brighter area, looks blackish [18][19].

It proves that such trade-off can take place not only between lightness and brightness but also between other corresponding pairs of material and lighting colour dimensions [20]. Indeed, Eqs 30 can be transformed as

$$A \int_{\lambda_{\min}}^{\lambda_{\max}} \exp\left[-(\theta_m+\theta_l)\left(\lambda - \frac{\theta_m\mu_m+\theta_l\mu_l}{\theta_m+\theta_l}\right)^2\right] \times s_i(\lambda)\, d\lambda = \varphi_i(xp), \quad (31)$$

where

$$A = k_m k_l \exp\left[-\left(\theta_m\mu_m^2+\theta_l\mu_l^2\right) + \frac{(\theta_m\mu_m+\theta_l\mu_l)^2}{\theta_m+\theta_l}\right].$$

One can show that for each triplet (k_m,θ_m,μ_m) there is not more than one triplet (k_l,θ_l,μ_l) such that the sextuplet $(k_m,\theta_m,\mu_m,k_l,\theta_l,\mu_l)$ meets Eqs 31; and vice versa, given a triplet (k_l,θ_l,μ_l) there is not more than one triplet (k_m,θ_m,μ_m) such that the sextuplet $(k_m,\theta_m,\mu_m,k_l,\theta_l,\mu_l)$ meets Eqs 31. Hence, there can be only one pair

$(g_m(\lambda;k_m,\theta_m,\mu_m), g_l(\lambda;k_l,\theta_l,\mu_l))$ that is colour equivalent to (x,p).

To look into the relationship between the material and lighting parameters (i.e., (k_m,θ_m,μ_m) and (k_l,θ_l,μ_l)) let us keep the parameters of the Gaussian under the integral sign in Eqs 31 constant, that is, let us assume that

$$\theta_m+\theta_l=\theta, \text{ and } \frac{\theta_m\mu_m+\theta_l\mu_l}{\theta_m+\theta_l}=\mu, \tag{32}$$

where μ and θ are some constants. In this case the left-hand-side terms will be proportional to the corresponding right-hand-side terms in (31). Therefore, if only the chromaticity is kept constant, that is, for $i=1,2,3$

$$\varphi_i(xp)/(\varphi_1(xp)+\varphi_2(xp)+\varphi_3(xp))=const,$$

then the trade-off between material and lighting components is described by Eqs 32 which can be rewritten as

$$\theta_m=\theta-\theta_l \tag{33}$$

$$\mu_m=\frac{\theta}{\theta-\theta_l}\mu-\theta_l\mu_l. \tag{34}$$

It follows from Eq. 34 that the parameters θ_m and θ_l are bound by a simple linear relation. The parameters μ_m and μ_l are also linearly related to each other. When θ_m and θ_l are kept constant μ_l decreases when μ_m increases, and vice versa.

While it has always been intuitively clear that the same light coming to the eye can be produced by, on the one hand, say, a red surface illuminated by a green light, and on the other hand, a green surface illuminated by a red light, there has been no quantitative formulation of this. Eqs 33 and 34 offer a theoretical framework for evaluation of these mutually interchangeable object/light pairs.

If the parameters μ_m and μ_l are considered, at least to the first approximation, as the stimulus correlates of the material and lighting hue respectively, then Eq. 34 indicates that there might be an invariance relationship between material and lighting hues similar to the invariance relationship between lightness and brightness mentioned above. Some colour illusions observed under pseudoscopic transformation of the apparent relief in real scenes support this conjecture [20].

From colour to colour image

Natural scenes usually contain a large number of fragments with different spectral reflectances illuminated by multiple light sources. The colour of each fragment can be described with two triplets of tri-stimulus values – the material and lighting ones (i.e., with the sextuplet of six-stimulus values). It should be noted that the lighting tri-stimulus values are generally different for different fragments even if the scene is illuminated by a single light source. This is a direct consequence of the inseparability of the colour equivalence. Indeed, if the colour image of a single-illuminant variegated scene could be described as a spatial distribution of six-stimulus values, the lighting component of which were constant, it would amount to that an object-colour can be specified by the three numbers which are determined by the illuminant only, and the three numbers solely determined by the spectral reflectance. This would be possible only if colour equivalence can be separated into two equivalence relations independently determined on the light and object sets. However, such a separability is impossible because of metamer mismatching.

One might argue that our experience testifies that we seem to never experience more illuminants than in reality. Particularly, a single-illuminant scene is usually perceived as a single-illuminant scene. It should be noted, however, that the variance of lighting tri-stimulus values for a single-illuminant scene tells us nothing of the way the human visual system encodes colour. It simply means that employing only a single light from the lighting colour atlas, one cannot simulate (i.e., produce a match for) the colour of an arbitrary reflecting object lit by an arbitrary single light. Figuratively speaking, a single-light representation of a single-illuminant scene is, generally, impossible. As with classical colorimetry, the present theory can predict whether two object-colour stimuli will look alike, but it does not tell us what they will look like.

Still, a single-light representation can prove to be a useful approximation to the true representation of a single-illuminant scene. Consider, for example, a set of six-stimulus values produced by a single-illuminant scene. As it is a single-illuminant scene, one can expect that the variation of the lighting tri-stimulus values will be not large. In this case one can try to find a single element of the light colour atlas (i.e., a single lighting tri-stimulus value) that brings about the material tri-stimulus values which minimally deviate from the true material tri-stimulus values. This will be referred to as the minimal single-light representation of a single-illuminant scene. Of course, one needs a criterion of "minimal deviation" that, in turn, implies some measure of proximity in the object colour atlas. As the first approximation one can use the chromaticity difference based on the spherical metric [8].

Note that the element of the light colour atlas which brings the minimum deviation can only incidentally coincide with the one that is metameric to the actual illuminant. Therefore, knowledge of the XYZ tri-stimulus values of the illuminant does not yield the minimal single-light representation of a single-illuminant scene. In other words, the problem of the minimal single-light representation of a single-illuminant scene cannot be reduced to that of estimation of the illuminant XYZ tri-stimulus values.

As there is general belief that illuminant estimation is a key component of the colour computation performed by the visual system [21][22], let us look into the issue in more detail. Consider a single-illuminant, variegated scene. In such a scene each pixel determines three equations as Eqs 29. The resultant colour distribution in the scene can be considered a result of resolving this system of simultaneous equations. Although each pixel in such a scene can, in principle, be assigned a colour with different lighting coordinates (q_n,q,μ), it is highly unlikely. It seems plausible to assume that the visual system tries to find a solution to the system of simultaneous equations (29) with as small number of light sources as possible. First of all it might attempt at solving the system of simultaneous equations (29) under the assumption of a single illuminant. In this case equations (29) have to be solved with respect to the material coordinates α,δ, and $\overline{\lambda}$ for a fixed triplet of the lighting coordinates q_n,q, and μ. Assume also that the visual system performs an illuminant estimate first. Suppose, for example, that the lighting coordinates q_n,q, and μ of the illuminant are derived from the XYZ tri-stimulus values of the perfect reflector. Specifically, replacing $r\left(\lambda;\alpha,\delta,\overline{\lambda}\right)$ in Eqs 29 with the spectral reflectance function taking 1 at every wavelength we have $(i=1,2,3)$

$$\int_{\lambda_{\min}}^{\lambda_{\max}} g(\lambda;q_n,q,\mu)s_i(\lambda)d\lambda=\varphi_i(p). \tag{35}$$

Thus, if along with the XYZ tri-stimulus values of the light reflected by the object, that is, $(\varphi_1(xp), \varphi_2(xp), \varphi_3(xp)))$ the XYZ tri-stimulus values of the illuminant (i.e.$(\varphi_1(p), \varphi_2(p), \varphi_3(p)))$ are known, then we have a system of six equations with six unknown parameters that can be resolved uniquely. Moreover, solving equations (35) with respect to q_n, q, and μ, one can determine the Gaussian $g(\lambda; q_n, q, \mu)$ colour equivalent to the illuminant p, and then substituting it into Eqs 29 one can resolve these thereby determining the spectral reflectance function $r\left(\lambda; \alpha, \delta, \overline{\lambda}\right)$.

Note, however, that the solutions obtained this way will differ from the true solutions (i.e., those obtained by using the spectral power distribution of the illuminant and the spectral reflectance of each pixel) not only in the lighting but also in the material tri-stimulus values. It is to be investigated the difference between the material six-stimulus values computed by using the illuminant XYZ tri-stimulus values and the true material tri-stimulus values.

It must be said that it is not the case that for every set of XYZ tri-stimulus values one can resolve the system of simultaneous equations (29) with respect to the material coordinates α, δ, and $\overline{\lambda}$ for a fixed triplet of the lighting coordinates q_n', q', μ'. A necessary condition for this is that the XYZ tri-stimulus values $(\varphi_1(xp), \varphi_2(xp), \varphi_3(xp))$ for every pixel make a configuration in the XYZ colorimetric space which can be inscribed in the object-colour solid determined by the Gaussian $g(\lambda; q_n', q', \mu')$. Therefore, for single-illuminant scenes well articulated with respect to reflectance, the exact solution of equations (29) with respect to α, δ, and $\overline{\lambda}$ for a fixed triplet q_n', q', μ' is rather unlikely to exist. In this case an approximate solution (i.e., one based on the minimal single-light representation) can be an alternative.

Conclusion

It has long been recognised that in order to represent the colour of a reflecting object illuminated by various lights one needs at least six numbers. All modern models of colour appearance take as an input the XYZ tri-stimulus values of the incident light as well as the XYZ tri-stimulus values of the light reflected from the object [23]. Although six numbers is enough to specify object colour, they cannot be derived from these two tripletes of the XYZ tri-stimulus values. Moreover, all these models have only three independent output variables despite the fact that, technically, the number of colour appearance dimensions predicted by the models can be more than three. An implicit assumption behind all these models is that object colour can be described as a three-dimensional manifold independent of the illuminant. However, the object-colour manifold is shown here to be a six-dimensional manifold. Object-colour requires six numbers (namely, the six-stimulus values introduced above) for its specification. Even in a single-illuminant, variegated scene the object-colours cannot be specified with only three numbers.

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