

Riemann Geometry for Color Characterization and Mapping

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Abstract

In this paper, we use Riemann geometry to develop a general framework for the characterization of and mapping between color spaces. Within this framework we show how to construct maps, so-called isometries, between two color spaces that preserve color differences. We illustrate applications of this framework by constructing a uniform color space and developing algorithms for color reproduction on different printers and correction of color-vision for color-weak observers.

Introduction

Many traditional color science methods are based on statistical averages. The most well-known is the standard observer describing the basic color vision properties of an observer with average color vision. This is useful in applications like the evaluation of color reproduction properties in the printing industry where comparison of methods or products on average are required.

It is less useful where applications require more exact descriptions of individual properties or where the target is another device like a printer. Obviously methods using the standard observer fail completely for problems involving color blindness. For these applications more precise, tailored procedures are necessary. Similar problems are handled in the framework of color management systems. But methods used there are of limited use when human observers are the "devices" to be handled since it is impossible to "calibrate" a human observer by measuring input/output value pairs.

Thus, in the widest sense of color reproduction or color management, one has to consider mappings between various devices and individual human observers. In particular, one of the most natural, and therefore useful maps, is a map which preserves subjective color-differences. In the language of Riemann geometry, such a map is called an isometry.

In general it is non-trivial to construct an isometry. In fact, a map which preserves small color-differences around one point can be easily obtained by linear algebra, since locally the map can be represented by its piece-wise linear approximation at the point, which can be found using the local isometry condition, or simply matching the threshold ellipses/ellipsoids at the preimage point and the image point.

However, a map preserving color-differences at every point (an isometry) between two spaces and a global isometry or a map which preserves large color-differences are usually more difficult to construct since the above piece-wise linear approximation should take a form of an affine transformation in global coordinates. The affine or parallel shift information after the mapping is not easy to come by. Another, and more serious problem, is that the map between color spaces are usually assumed to be monotonic for simplicity such as in the ICC profiles. When the map is not monotonic the inverse map is not injective or one-to-many and thus hard to find.

In this paper, we introduce Riemann geometry as a tool to

compute geometrical descriptions of color spaces from measurement data and mapping between color spaces.

The input data are descriptors of the local properties of the color space under investigation. The best known examples are the MacAdam ellipses. (see [1], [2], [3] and [4]). Other examples are measurements of the color coordinates of a printed test chart or descriptors computed from other color systems.

We first show how to compute an isometry between a color space and the Euclidean space. From a geometrical point of view, a coordinate system is the most straightforward characteristic of a color space. Another characteristic object of a geometrical space are its straight lines, or geodesics in Riemann spaces. From local threshold measurements the geodesics can be calculated by solving differential equations known as the geodesic equations. We will show how to obtain a "polar coordinate system" in a color space using the geodesics. The correspondence of this system with the polar coordinate system in the Euclidean space then provides us an isometry between the two spaces. As an example of this method we show the construction of uniform space for CIELUV.

Then we show how to build an isometry between two color spaces based on the above method. In particular, once the geodesic polar coordinate systems are obtained for these two color spaces, one can easily read out the isometry between these two color spaces.

As the first application of the framework, we show a color reproduction on two printers, MP790 by Epson and PM970C by Canon. Here the isometries between the working color spaces of different devices provide us a way to handle device-independent color reproduction.

Apart from investigations involving color blindness it is often ignored that there is also a variation in the color vision properties between observers with full color vision. Problems involving color weakness have received little attention in color technology. As the second applications, we will therefore illustrate how to characterize the color spaces of color normal observers and a color weak observer. An exact correction method is obtained using an isometry between this two color spaces called a color-weak map.

The framework described here has also applications in other fields. In [5], for example, the related model of a fibre bundle of the color spaces was used in image processing.

Riemann Geometry for Color Spaces

Intuitively a Riemann space is a generalization of Euclidean flat space where the distance between two points is measured in the same way everywhere. An intuitive way to understand how this can be generalized to a Riemann space is to consider a surface S like the unit sphere. Locally a neighborhood of a point x on S looks like a plane and can be approximated by the tangent plane $T_x S$. On this tangent plane we have a positive definite matrix $G(x)$ defining the scalar product between tangent vectors $u, v \in T_x S$ as $u^T G(x)v$. This scalar product defines an arc-

length on the Riemann space in the same way as in Euclidean geometry.

A formal definition of a Riemann space is a space S with a positive definite matrix $G(x)$ smoothly defined on every point x such that the infinitesimal distance near x is measured by $(dx, dx)_G = dx^T G(x) dx$. Let $x = (x^1, \dots, x^n)^T$, $G(x) = [g_{ij}]$, then

$$(dx, dx)_G = g_{ij} dx^i dx^j$$

(where the Einstein convention is used to sum over indices that appear as both subscripts and superscripts as $a^i b_i = \sum_i a^i b_i$). Such a Riemann space is usually denoted as $S = (\mathbb{R}^n, G)$. The global distance of any two vectors x, x' in S is measured as follows. If a spatial curve $x(t) = (x^1, \dots, x^n)^T$ in a Riemann space (\mathbb{R}^n, G) is smooth for $a \leq t \leq b, x(a) = x, x(b) = x'$, i.e., if $\dot{x}^i = \frac{dx^i}{dt}$ exist and are continuous, then the integral

$$l = \int_a^b \sqrt{g_{ij} \dot{x}^i \dot{x}^j} dt \quad G = (g_{ij}) \quad (1)$$

exists, and l is the length of the curve between $x(a) = x$ and $x(b) = x'$.

In Euclidean geometry the ordinary straight lines are the shortest curves connecting points. The corresponding shortest curves in a Riemann space are the geodesics. The distance $d(x, x')$ between x and x' is defined as the minimum of the above lengths among all possible smooth curves connecting the two points. The curve $\gamma(t)$ that has the shortest length is called the geodesic between the two points.

If we denote the elements of the matrix G by g_{ij} and the elements of the inverse matrix G^{-1} by g^{ij} then it can be shown that the geodesic $\gamma(t) = (x^1(t), \dots, x^n(t))^T$ are solutions of the differential equations:

$$\ddot{x}^k(t) + \dot{x}^i(t) \dot{x}^j(t) \Gamma_{ij}^k(x(t)) = 0 \quad (2)$$

where $\dot{x}^k(t) = \frac{dx^k(t)}{dt}$ and

$$\Gamma_{ij}^k(x) = \frac{1}{2} g^{km} \left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right). \quad (3)$$

The $\Gamma_{ij}^k(x)$ are known as Christoffel symbols. Once we know the Riemann metric G , the geodesics are obtained using a numerical differential equation solver.

Obtaining a Riemann metric

We saw that the geometry is completely defined by the matrices $G(x)$ and if we want to apply methods from Riemann geometry then we need access to these matrices. We will use two methods to obtain the Riemann metric of a color space: direct measurement of thresholds and measurement of Jacobians of color mappings.

Using measured threshold data

In a typical application we use color matching experiments to measure the just-noticeable-difference (jnd) of an observer in different directions from N points $x_v, v = 1 \dots N$ in a color space C . Using these measurements we fit an ellipsoid and use this ellipsoid to define the local metric $G(x_v)$. Assume at point x the jnd threshold ellipses are with the long axis a , the short axis b and the angle θ from the long axis to the axis of abscissa. Then the Riemann metric matrix G is given by

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{with}$$

$$g_{11} = \frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{a^2 b^2}$$

$$g_{12} = g_{21} = \frac{-2 \sin \theta \cos \theta (a^2 - b^2)}{a^2 b^2}$$

$$g_{22} = \frac{b^2 \sin^2 \theta + a^2 \cos^2 \theta}{a^2 b^2}$$

Using Jacobian of a color map

For applications where we have a n -D color space C with Cartesian coordinates, which can be regarded as an Euclidean space \mathbb{R}^n . A map f of C which maps a color vector x to a color vector $y = f(x)$ induced a Riemann metric of C as

$$G(x) = D_f(x)^T D_f(x) \quad (4)$$

where $D_f(x)$ is the Jacobian matrix of f at the point x . This metric then defines $C = (\mathbb{R}^n, G)$ as a Riemann space.

Color-Difference Preservation: Isometries

An isometry is a map between two Riemann spaces that preserves distances between two points. Isometries between color spaces preserve color-differences and this is the reason why they play an important role in color science and engineering.

We introduce two definitions of an isometry, one local and another global. We define a local isometry as a map between color spaces which preserves the jnd threshold or the Riemann metric at every point

$$f : C_1 \xrightarrow{\sim} C_2 \quad (5)$$

$$x \mapsto y \quad (6)$$

$$G_1(x) = D_f^T(x) G_2(y) D_f(x) \quad (7)$$

where G_1, G_2 are the Riemann metrics of C_1 and C_2 .

A global isometry, on the other hand, is a map which preserves all color differences

$$f : C_1 \xrightarrow{\sim} C_2 \quad (8)$$

$$x \mapsto y \quad (9)$$

$$d_1(x, x') = d_2(y, y') \quad (10)$$

where d_1, d_2 are the distances in C_1 and C_2 .

It can be shown that a local isometry is also a global isometry and vice versa[6]. This means that if one can preserve the threshold ellipses or ellipsoids everywhere as in (7), then large color differences will also be preserved.

Uniformization: Isometry onto Euclidean Spaces

The simplest Riemann space is a Euclidean space where the Riemann metric defines the Euclidean distance. We will simplify the construction of isometries between color spaces by first considering the case where one space is a Euclidean space: $u : C = (\mathbb{R}^n, G) \rightarrow U = (\mathbb{R}^n, I_n)$. In terms of color engineering, such a space with Euclidean metric is called uniformation color space (UCS), and we can think of it as a connection space. We will call this isometry a uniformization map of C .

In [6] it was shown how, using a geodesic grid, one can build a uniformization map. This algorithm works as follows. First choose a point as the origin O , then draw geodesics emanating from O , with equal angle increments measured by the Riemann metric at O . Next connect the points of the same distance from O on the neighboring geodesics. This gives a curvilinear coordinate grid on the Riemann space similar to the coordinate

system on the sphere with the north (or south) pole and the longitude/latitude coordinates. This generalized polar coordinate system has the geodesics as the radius lines and the equal-distance lines as the concentric circles. These coordinates are called Riemann coordinates or normal coordinates.

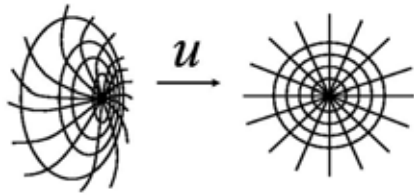


Figure 1. Mapping from color space to Euclidean space

The isometry between the color space and the Euclidean space, as a coordinate transform, can be read out directly from the geodesic chart of the color space. In particular, a point in the color space with the Cartesian coordinates e.g. $P = (s, t)$ corresponds to the point in the Euclidean space with polar coordinates $P' = (\rho, \theta)$ where ρ is the geodesic distance from P to the origin O and θ is the angle between the geodesic of zero degree to the geodesic from P to O . The computation of the inverse map is obviously also trivial.

Example: uniformization of CIELUV

Here we illustrate this strategy computing the uniformization of CIELUV space. We construct UCS of CIELUV based on the MacAdam ellipses. The origin is chosen as the white point under D65. The results are shown in Figures 2 and 3. In Figure 2 we plot the geodesic grid as the Riemann coordinates. We apply the coordinate transform from the geodesic coordinates in the CIELUV space to the polar coordinates in the UCS thus establishing the uniformization map from CIELUV to the UCS. In Figure 3, the threshold ellipses as the images of the MacAdam ellipses after uniformization are plotted also in CIELUV. It can be observed that local uniformity of the space has been improved significantly.

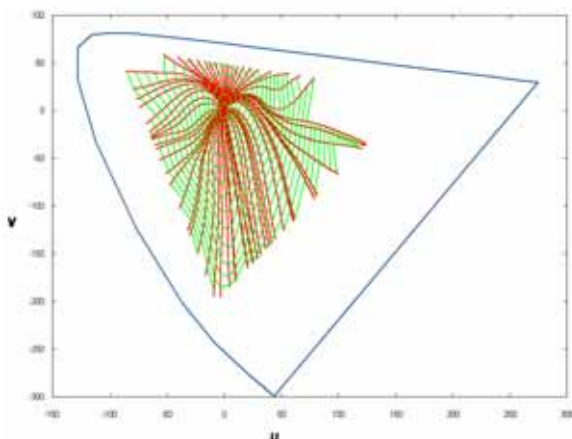


Figure 2. Riemann coordinates of the CIELUV space

Building Isometries between Color Spaces

Using the Riemann coordinates of a color space, introduced above, we can explicitly build an isometry between any two color spaces.

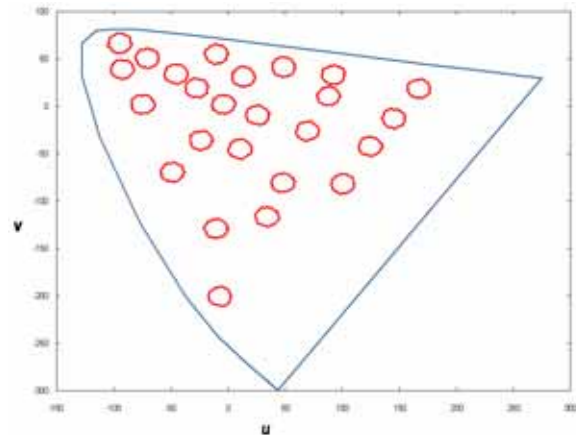
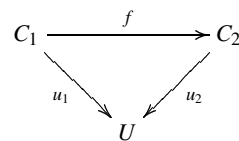


Figure 3. Thresholds of the CIELUV after uniformization

Assume we have two color spaces, which could be a color space of a device or the perception color space of a human observer.

Let C_1 be the first color space, we denote it as Riemann space (C_1, G_1) , C_2 the second color space denoted as Riemann space (C_2, G_2) , U is an Euclidean space and its Riemann space representation is (U, I) with I as the unit matrix, $u_1 : C_1 \rightarrow U$ the uniformization map of C_1 , $u_2 : C_2 \rightarrow U$ the uniformization map of C_2 .

An isometry $f : C_1 \rightarrow C_2$ between C_1 and C_2 can then be constructed as shown in the following commutative diagram.



The map constructed as

$$f : C_1 \rightarrow C_2 \tag{11}$$

$$f = u_2^{-1} \circ u_1 \tag{12}$$

is an isometry since u_1, u_2 are isometries so is their composition.

The isometry between two different color spaces is obtained as follows: First a pair of corresponding points, one from each space, is selected. These points are selected as the origins in each color spaces. In fact, a different choice of the origin will introduce a parallel shift in the coordinate transform. Next a geodesic coordinate system in each of the color spaces is constructed by using the geodesic grids from the previous section. This gives two coordinate systems similar to the polar coordinate system in the plane. Using these two coordinate systems we can map points in one system to corresponding points in the other system. In fact, the isometry can be computed very easily. Since every point in each space has its own Riemann coordinates already given, to look for its image at the other space, one only needs to look for the point in the other space which has the same Riemann coordinates.

Application to Color Reproduction

Here we use two printers: an PM-970C by Canon and an MP790 by Epson. To derive the Riemann metrics, first we use

the color samples shown in Fig. 4 in their CIELAB coordinates. Measurements of the outputs of the color-samples by the two printers are plotted in CIELAB space in Fig.5 and Fig. 6. These outputs are used to estimate the Jacobian matrices of the maps from the input space to the color spaces of each printer at the measured points. Then Riemann metrics for these two color spaces are obtained using the Jacobian matrices as shown in (4).

Next the Riemann coordinates of both spaces are obtained using geodesic grids. The simple example of coordinate systems on the sphere shows that it may not be possible to construct one global coordinates for the whole Riemann space. It is therefore often necessary to use several local but compatible coordinates in parallel. Besides that, parallel computations are desirable for fast implementations. We used a multi-start strategy to build the Riemann coordinates. These local coordinate systems are shown in Fig. 7 and Fig. 8. Afterwards an isometry between these color spaces is obtained.

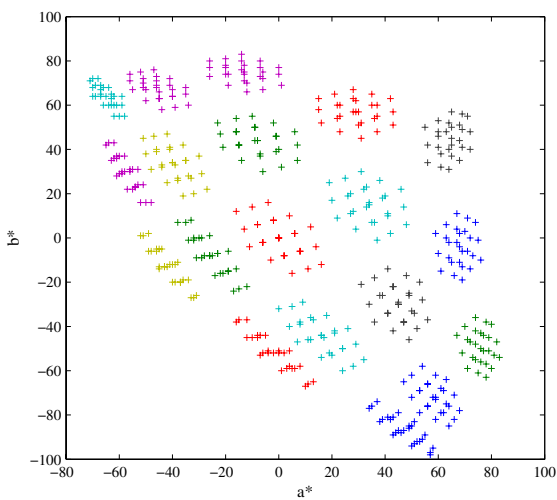


Figure 4. Color-samples in CIELAB to estimate Riemann metric

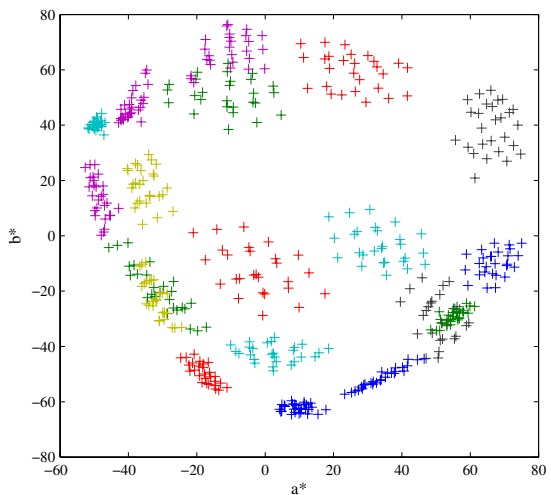


Figure 5. Measurement of color-samples output by PM970C

We illustrate an application of this isometry by showing how to simulate the output of one printer on a second printer. The goal is to reproduce the output of the MP790 on the PM-970C. As a color-target used for color reproduction we use the colors shown in Fig. 9. Printing these colors on the two printers results in the outputs in Fig. 10. Their CIELAB coordinates are plotted

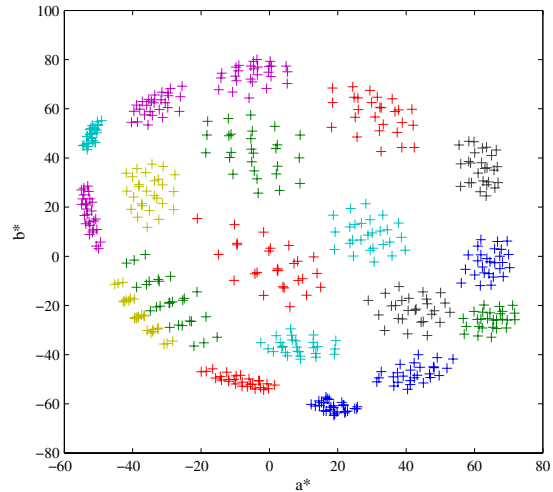


Figure 6. Measurement of color-samples output by MP790

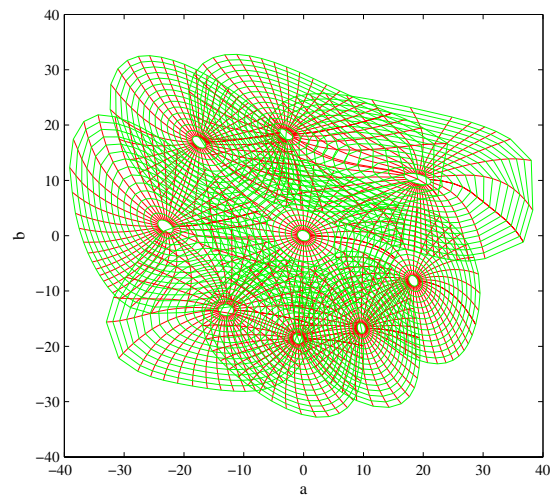


Figure 7. Riemann coordinates system of PM-970C in CIELAB

in Fig. 11. It can be observed that the outputs of two printers of the same color-targets are quite different.

To test color reproduction we now fix the output of the MP790 and reproduce its output on the PM-970C. In order to do that, the input to PM-970C is transformed by the inverse map of the above isometry. The reproduction of the output of MP790 by PM-970C is shown in Fig. 12. Measurements of all three outputs, one of MP790, two of PM-970C before and after applying the inverse isometry are plotted in CIELAB Fig. 13. The output of PM-970C with its input transformed produced much closer color reproduction of MP790 than without.

Application to color-weak correction

Now we show an application of the proposed framework for correction of color-weak observers[8].

The jnd thresholds for a color-weak observer and an average of 45 color-normal observers are shown in Fig. 14 and Fig. 15 respectively.

The Riemann coordinates for the color spaces of color-normal and color-weak observers are obtained by geodesic grids with the origin as white-point under D65. They are shown in

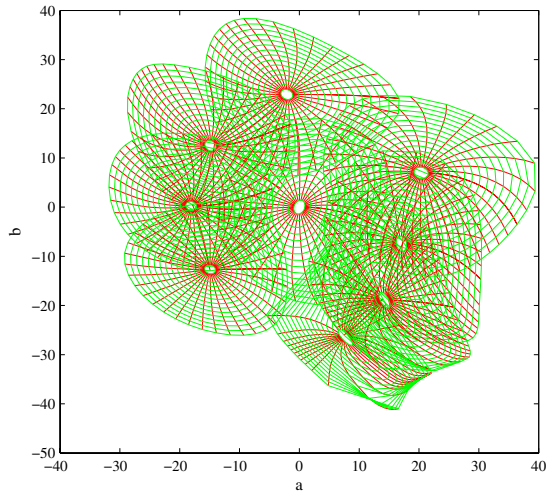


Figure 8. Riemann coordinates system of MP790 in CIELAB



Figure 9. Color targets used for color-reproduction

Fig. 16 and Fig. 17 respectively.

The isometry defined from the color-normal space to the color-weak space is called a color-weak map [8].

Applying this color-weak map to input images, color-normal observers can experience the simulated images which the color-weak observer actually perceive. On the other hand, applying the inverse map of the color-weak map to the images before showing to the color-weak observer, one can present to him the same color stimuli perceived by color-normal observers.

Conclusions

We introduced Riemann Geometry as a tool to quantify properties of color spaces of individual observers or devices. In particular, the framework is used to show how to create color-difference preserving mappings between different color spaces in the form of geodesic coordinates. This framework is applied to build a uniform color space of CIELUV space, to improve color reproduction between two printers and correction and simulation of color vision for color-normal and color-weak observers.

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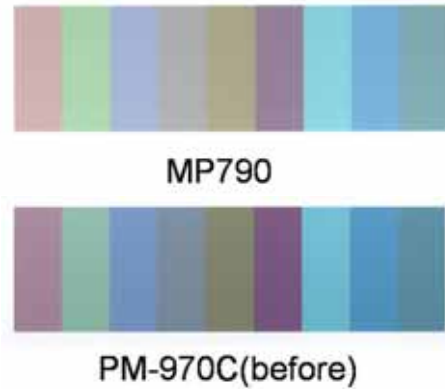


Figure 10. Output of PM970C and MP790

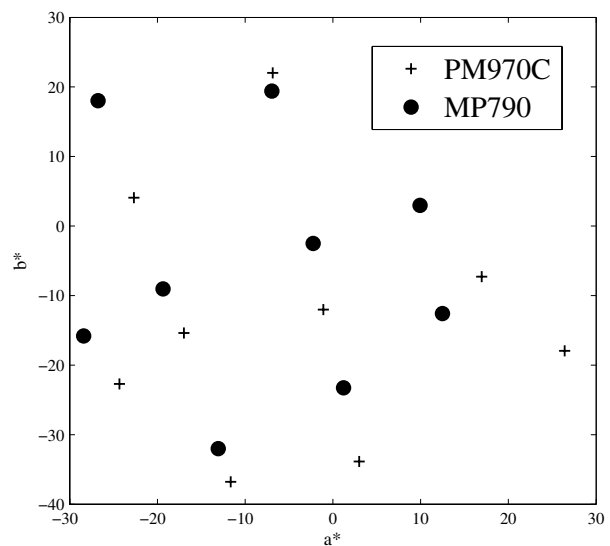


Figure 11. Output measurement of PM970C and MP790 in CIELAB

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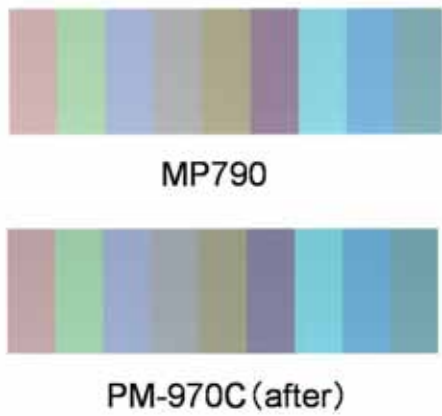


Figure 12. Reproduction of output of MP790 by PM970C

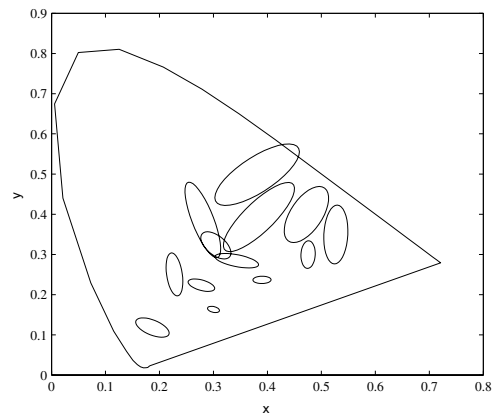


Figure 15. Threshold ellipses of a color-weak observer

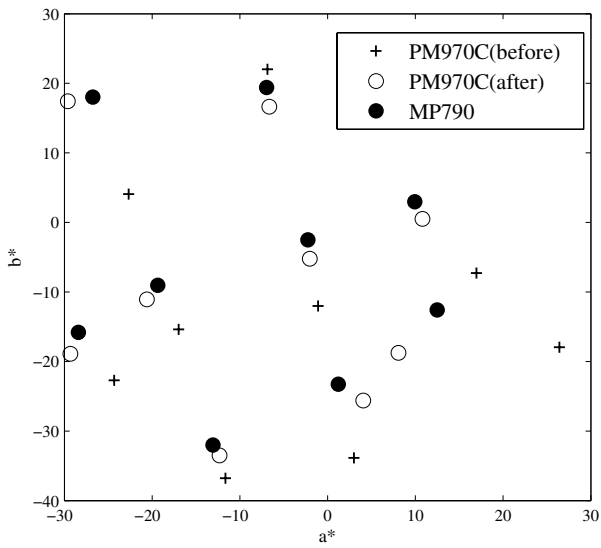


Figure 13. Output measurement of MP790 and reproduction by PM970C

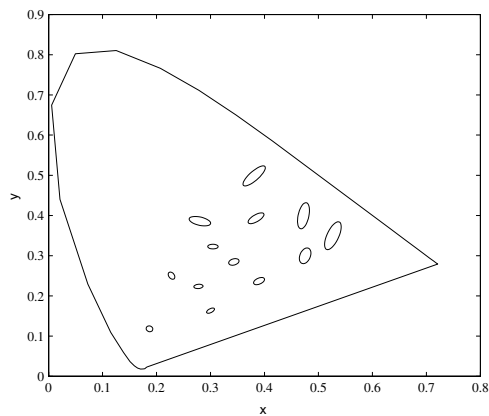


Figure 14. Threshold ellipses of color-normal observers

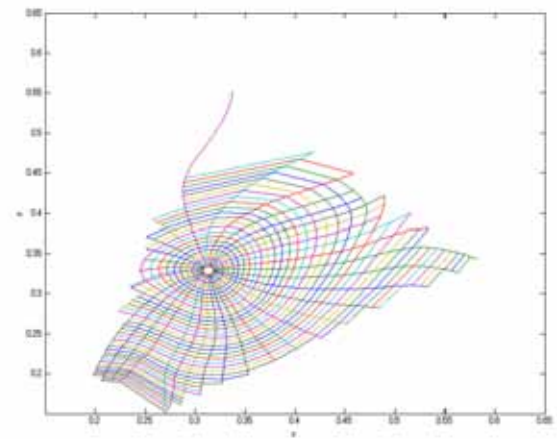


Figure 16. Riemann coordinates for observers with normal color vision

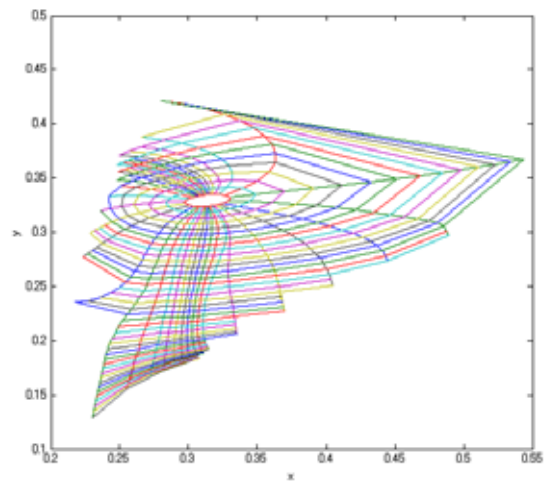


Figure 17. Riemann coordinates for an observer with color-weak vision