Time-Frequency Analysis of Color Spectra

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Abstract

Most physics-based color processing methods define color signals as functions of wavelengths. Such a description characterizes the color signal completely from a physical point of view. This representation has however several drawbacks: the description is redundant and important properties are often hidden and difficult to extract from this representation. Other color representations may provide more compact descriptions or descriptions in which the relevant properties are more clearly visible. Other color systems have thus been used, including popular color signal representations based on principal component analysis or Fourier transformations. Most of these methods decompose color signals as linear combinations in a given basis system and they require thus the selection of a basis before they can be applied. In this paper we introduce time-frequency methods as an alternative that avoids the selection of a basis and provides a signal representation that combines advantages of both, the wavelength and the Fourier transform based signal description.

1. Introduction

Traditional descriptions of color (such as RGB, CIEXYZ, CIELAB or HSV) are almost all three-dimensional. For most applications this is not a problem since the human color vision system is also based on three types of sensors. For some problems this is however not sufficient since too much information is lost when the information in a spectral distribution is coded into only three numbers. Typical examples where better descriptions are needed are applications where the interaction between illumination, reflection and detector is important. Here the full spectral information is usually necessary. The need for better color descriptors has led to the establishment of the new research field of multichannel color processing including the development of color imaging sensors with higher spectral resolution and new models for investigating sets of color spectra.

Almost all models used in multichannel color processing select a basis in the space of spectra, expand a given spectrum in this basis and take a few coefficients as coordinates. Popular choices of the basis are Principal Components, Fourier series or even polynomial systems (a few examples are [1, 2, 3], see also [4]). All of these descriptions have to select a basis first! A basis that is well-adapted to one class of spectra may not be useful for another set of spectra. Also different applications might require different types of bases. A class of signal representations that avoid many of these problems are Time-Frequency Distributions (TFD) of which the Wigner distribution is the oldest and most popular. In this paper we give a brief introduction into TFD and describe why it is relevant to multi-channel color processing. In a companion paper TFD are used to study the interaction between illumination and reflection and we will illustrate how time-frequency distributions can be used to solve pattern recognition tasks.

Motivation

Time-frequency methods are by now a well-established research field in signal processing [5, 6, 7] with its own terminology. In this paper we will mainly use this terminology with the exception that we will not use the term time but talk of wavelength or position instead. Some confusion may arise because the term spectrum has different meanings in color science and signal analysis and mathematics. We hope that this will not be a problem and that the appropriate meaning will be clear from the context.

As an illustration of the basic ideas behind the application of TFD's in color processing consider the spectra shown in Figure 1.

The first spectrum comes from a PolyluxXL/860 lamp, the second from a TLD18/16Yellow lamp and the third is a daylight spectrum measured in Norrköping, Sweden. In signal processing this representation is known as the time-signal. Here we call it the wavelength signal. Another basic form of signal representation is the frequency representation in which the signal is represented as a superposition of elementary sine and cosine waves. The contributions of the different elementary sines and cosines is given by the coefficients of the Fourier transform defined as

$$S(\xi) = \frac{1}{\sqrt{2\pi}} \int s(\lambda) \mathrm{e}^{-i\lambda\xi} \, d\lambda \tag{1}$$

We will usually denote Fourier transforms with capital letters: S is thus the Fourier Transform of s.

We think of the wavelength representation and its Fourier transform as representations of the same color signal in dif-

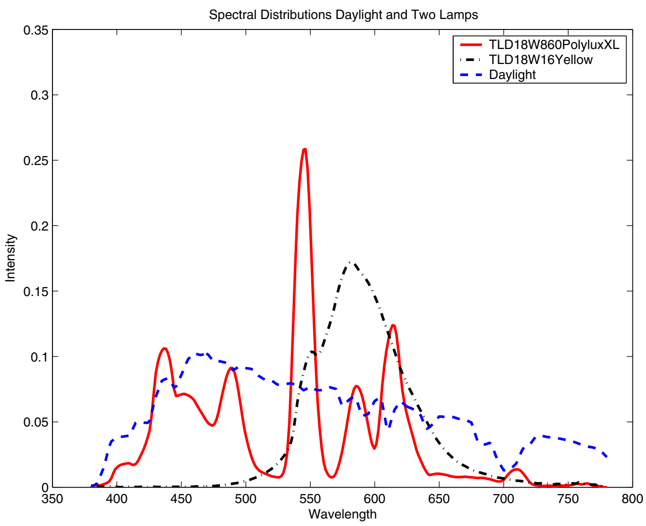

Figure 1: *Spectral Distributions of Daylight and two Lamps*

ferent coordinate systems since the signal can be recovered from its Fourier transform. Figure 2 shows the absolute value of the Fourier transforms of the illumination spectra shown in Figure 1.

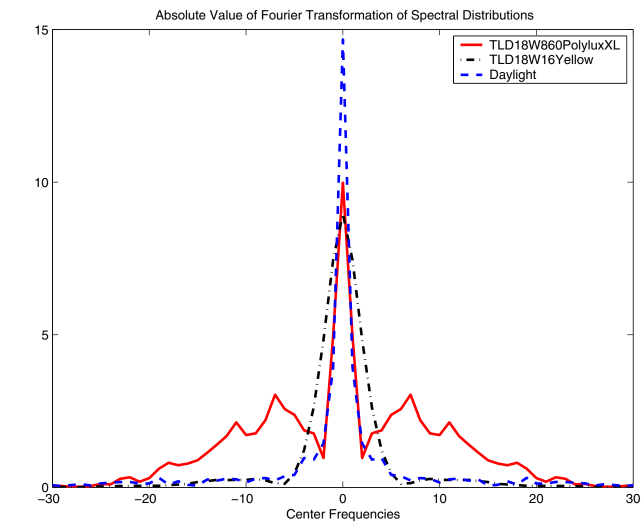

Figure 2: *Absolute Values of Fourier Transforms*

From the two Figures we see that the two representations illustrate different, characteristic features of the two sources. For example we see easily from Figure 1 that the peak of the Yellow lamp spectrum lies to the right of the peak of the corresponding spectrum for the PolyluxXL lamp. This information is not as easy to extract from the Fourier transform. On the other hand we see from Figure 2 that the Yellow lamp has a much smoother distribution than the PolyluxXL which has more contributions from higher Fourier coefficients. In the extreme case of a monochromatic spectrum the wavelength description is optimal since we can describe the spectrum completely with a single parameter, the location of the monochromatic contribution. The absolute value of the Fourier transform of such a mono-chromatic spectrum is however constant everywhere. The other extreme is a perfectly white light with a constant value in the wavelength description of the spectrum and a single peak in the Fourier description. This illustrates a general property of all signals: If they are concentrated in the wavelength representation then they are spread out in the Fourier domain and the other way around, if they are localized in the Fourier domain, they are spread out in the wavelength domain. In physics this is known as the uncertainty principle: A color signal cannot be concentrated in both the wavelength and the Fourier domain. In Figure 2 we see thus that the Fourier transform of the daylight spectrum is much more concentrated around the origin than the two lamps. These are only some observations to show that both signal representations give complementary descriptions of a signal and that for a more complete description a combination of these two is necessary.

Definition and Computation

An attempt to construct a signal representation that combines both the time and the frequency distribution aspects is Time-Frequency-Analysis. The oldest tool is the Wigner-Ville Distribution (WVD) introduced by Wigner in 1932. For a color spectrum $s(\lambda)$ this distribution as defined as:

$$W_s(\lambda,\xi) = \frac{1}{2\pi}\int s\left(\lambda+\frac{\tau}{2}\right)s\left(\lambda-\frac{\tau}{2}\right)\mathrm{e}^{-i\tau\xi}\,d\tau \quad (2)$$

The following observations may help to understand some properties of the transform:

- The main characteristic feature of the transform is the fact that it depends on the **PRODUCT** of the signal, or more precisely on the product of the signal and a folded version of it. It thus analyses the overlap of two parts of the signal that are located below and above the wavelength λ under consideration. If a property holds both below the current wavelength and above it then the property has an influence on the current value of the transform at λ.
- The same reasoning holds also for properties in the Fourier domain since

$$W_s(\lambda,\xi) = \int \overline{S\left(\xi+\frac{\theta}{2}\right)}S\left(\xi-\frac{\theta}{2}\right)\mathrm{e}^{-i\theta\lambda}\,d\theta \quad (3)$$

- Another, obvious, observation is that the transform maps the one-dimensional (wavelength or Fourier) representation into a two-dimensional function W.

- The Wigner distribution is real and often it is useful to view it as a time-frequency energy or "probability" distribution: the value of $W_s(\lambda_0, \xi_0)$ at the point (λ_0, ξ_0) is the probability that there was a variation of frequency ξ_0 (in the Fourier sense) around the wavelength λ_0. This is not strictly true since it can be shown that for almost all spectra (except the Gaussians) the Wigner distribution will assume negative values. An evaluation of the WVD can be complicated but the probability interpretation is helpful in many applications.

Before we summarize a number of useful properties of the Wigner distribution we have to make two remarks about the implementation and computation of the transform. We first note that the Fourier transform $S(\xi)$ of a color signal is symmetric, i.e. $\overline{S(\xi)} = S(-\xi)$ since color spectra are real-valued functions. Using the Fourier transform as such would imply that the mean frequency (defined as $\int \xi |S(\xi)|^2 \, d\xi$) is zero. Also the spread of the frequencies (defined as $\int \xi^2 |S(\xi)|^2 \, d\xi$) would then be dominated by the distance between the contributions of $S(\xi)$ on the positive and negative frequency axis. The standard way to circumvent this problem is to replace the real spectrum s by the complex function z defined as

$$z(\lambda) = s(\lambda) + \frac{i}{\pi} \int \frac{s(\lambda')}{\lambda - \lambda'} \, d\lambda'$$

The Fourier transform $Z(\xi)$ of this new signal has now the property that it is zero for negative frequencies and has the same value as S for positive frequencies. The imaginary part $\frac{i}{\pi} \int \frac{s(\lambda')}{\lambda - \lambda'} \, d\lambda'$ is known as the Hilbert transform and the function z is the analytical signal. In the following we will thus always use the analytical signal instead of the original spectrum.

The Wigner distributions of the two lamps and the daylight spectrum is shown in Figures 3,4 and 5. All of these Figures show the first 30% of the frequency distributions and all of them use a logarithmic scale for the value of the Wigner-distribution. Comparing the distributions of the Yellow lamp and the daylight we see that the daylight is smoother (smaller frequency contents) and more blueish (shifted to the left, i.e. smaller wavelengths). We see also that the daylight spectrum contains some distinct higher frequency contributions in the middle and longer wavelength range. The PolyluxXL lamp has, as expected, many high-frequency components and a relatively complex Wigner distribution.

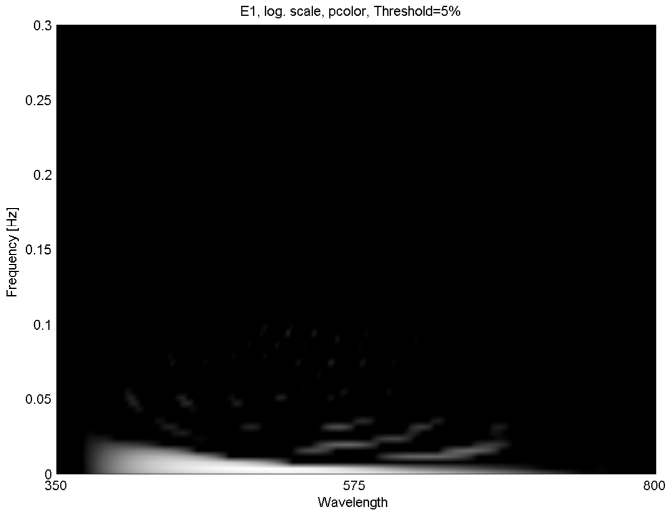

Figure 3: *Wigner distribution of the daylight spectrum*

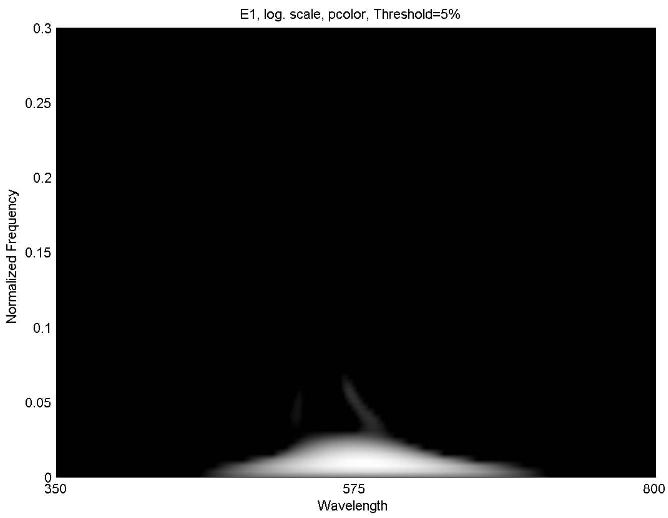

Figure 4: *Wigner distribution of the TLD18/16 Yellow Lamp*

Basic Properties

Among the many properties of the WVD we summarize here a few that are of obvious interest for color spectra:

1. If the spectrum s is zero outside the interval $[\lambda_m, \lambda_M]$ then the WVD $W_s(\lambda, \xi)$ (as a function of λ) is also zero outside this interval. A similar property holds also for the frequency part ξ.

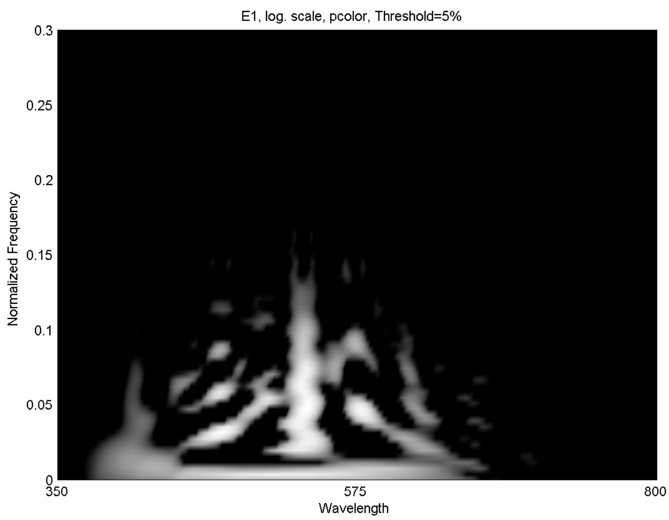

Figure 5: *Wigner distribution of the PolyluxXL/860 Lamp*

2. The signal can be recovered from its WVD:

$$s(\lambda) = \frac{1}{s(0)} \int W(\lambda/2, \xi) \mathrm{e}^{i\lambda\xi}\, d\xi$$

3. Shifting the spectrum results in a shift of the WVD (the same holds for frequency shifts)

4. The WVD is always real

5. It satisfies the marginals:

$$\int W_s(\lambda, \xi)\, d\xi = |s(\lambda)|^2, \text{ and}$$

$$\int W_s(\lambda, \xi)\, d\lambda = |S(\xi)|^2$$

(S is the Fourier transform of s)

6. It preserves the energy:

$$\int\int W_s(\lambda, \xi)\, d\lambda d\xi = \|s\|^2$$

7. Moyal relation: For two spectra s, s' we have:

$$\int\int W_s(\lambda, \xi) W_{s'}(\lambda, \xi)\, d\lambda d\xi = \int s(\lambda)s'(\lambda)\, d\lambda$$

8. Correct averages:

$$\int [g_1(\lambda) + g_2(\xi)]\, W_s(\lambda, \xi)\, d\lambda d\xi = \\ = \int g_1(\lambda)|s(\lambda)|^2\, d\lambda + \int g_2(\xi)|S(\xi)|^2\, d\xi \quad (4)$$

The simplest way to use Time-Frequency Analysis for color spectra analysis is the visual evaluation of the WVD. Since the WVD provides a representation of the signal that is similar to our interpretation in terms of position and frequency it is often easier to understand the WVD than the original spectrum.

Another way to utilize time-frequency analysis is related to the probability interpretation of the WVD. Even though it is not correct (remember, the WVD can have negative values) it can be used to compute probabilistic parameters that characterize the spectrum. As an example consider the correct averages property in Eq. 4. Using $g_1(\lambda) = \lambda, g_2(\xi) = 0$ gives a kind of mean value for the position probability distribution and $g_1(\lambda) = 0, g_2(\xi) = \xi$ the mean value for the frequency. Equation 4 gives also a recipe for other functions of λ and ξ. A similar measure of the location of the distribution is the coordinate (λ_M, ξ_M) where $|W|$ has its largest value (maximum probability). The first moments (mean values) of the wavelength and the frequency representation of 50 Planck spectra in the range 5000K to 12000K (with equal spacing in the Mired-parameter=inverse temperature scale) is shown in Figure 6. We see that the spectra are all very smooth (with the spectra at low temperature slightly smoother) and that the concentration of the distributions shift to the shorter wavelength (blue) region as the temperature increases.

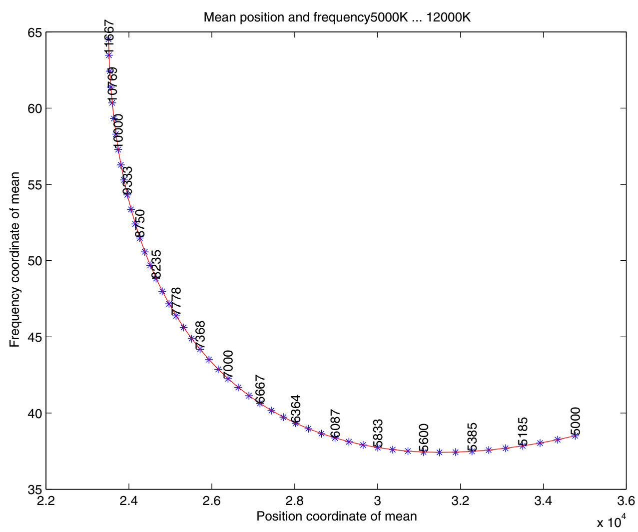

Figure 6: *Mean position and frequency values of the Wigner distributions of Planck spectra*

Filter design and basis pursuit

In Time-Frequency Analysis there are two important basic operations, Time-shifts and Frequency shifts. If s is a signal and λ_0, ξ_0 are the time and frequency shift parameters then the (time and frequency) shifted signal s_{λ_0,ξ_0} is defined as

$$s_{\lambda_0,\xi_0}(\lambda) = s(\lambda - \lambda_0)\mathrm{e}^{i\xi_0\lambda}$$

It can then be shown that shifting a spectrum in time and frequency results in a simple shift of the Wigner distribution.

The signals with the simplest TFD are scaled Gaussians since their shape is preserved under the Wigner transform. These Gaussian distributions are known as atoms. They are described by their width and their location on the wavelength and the frequency axes and they are characterized by being maximally concentrated in the TF-plane. Since time and frequency shifts of the signals result in shifts of the Wigner distributions it is natural to consider time-frequency shifted scaled Gaussians as the basic building blocks of the theory. It is therefore often desirable to approximate a given function as a linear combination of (position- and frequency-shifted) Gaussians. This gives a parameterized description of the original spectrum based on the parameters of the Gaussians used. The Moyal relation shows that matching can be done in the TF-plane or in the signal domain. Since the atoms are easiest described in the TF-plane it is natural to do the matching for the Wigner distributions. In the following example this is done by iteratively finding the best matching Gaussian atom and removing its influence from the current signal.

We illustrate the basic idea by analyzing the spectrum of the lamp PolyluxXL/860 used earlier in Figures (1,2,5). In the first series of Figures (7,8,9) we see how the different parts of the original spectrum are approximated by atoms.

In Figures(10, 11) the Wigner distributions of the approximations with two and eleven atoms are shown. This should be compared with the Wigner distribution of the original shown in Figure (5)

This experiment also demonstrates a fundamental difficulty with the TFD approach in general: The TFD is a quadratic representation of the signal and the TFD of the sum of two signals is not the sum of the TFD's of the signals but usually a complex interference pattern. The iterative algorithm used is thus only suboptimal. An optimal strategy should take into account the mixed terms in the TFD of the sum.

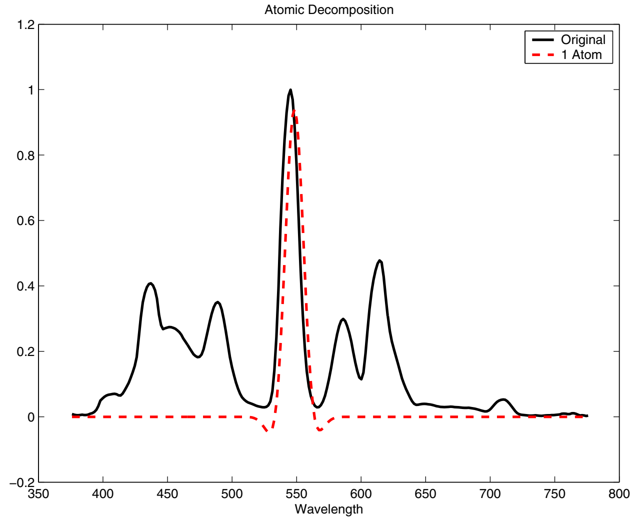

Figure 7: *Approximation of the spectrum of the PolyluxXL/860 lamp with one time-frequency atom*

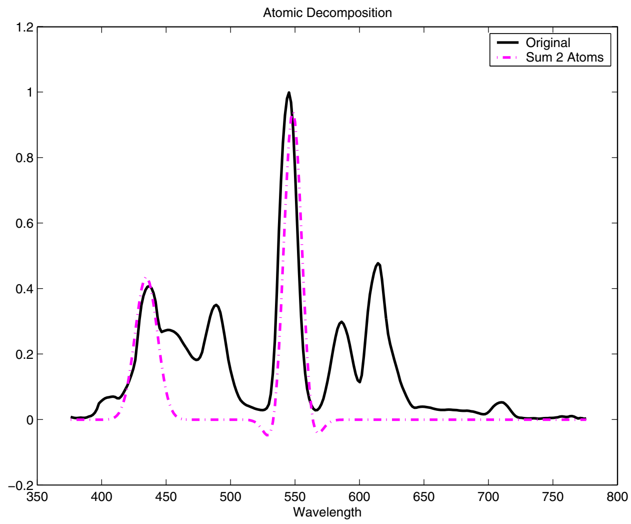

Figure 8: *Approximation of the spectrum of the PolyluxXL/860 lamp with two time-frequency atoms*

Conclusions

We gave an overview over some basic tools from the theory of Time-Frequency Analysis that should be of in interest in spectral based color processing. We showed that in this framework we get a description of the spectra that is independent of a selected coordinate system. As an illustration we showed how to generalize the approximation of spectra by Gaussians to approximations based on time and frequency shifted Gaussians.

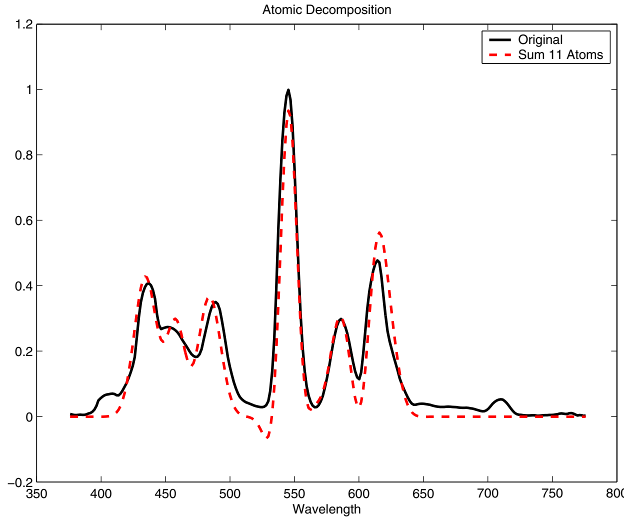

Figure 9: *Approximation of the spectrum of the PolyluxXL/860 lamp with eleven time-frequency atom*

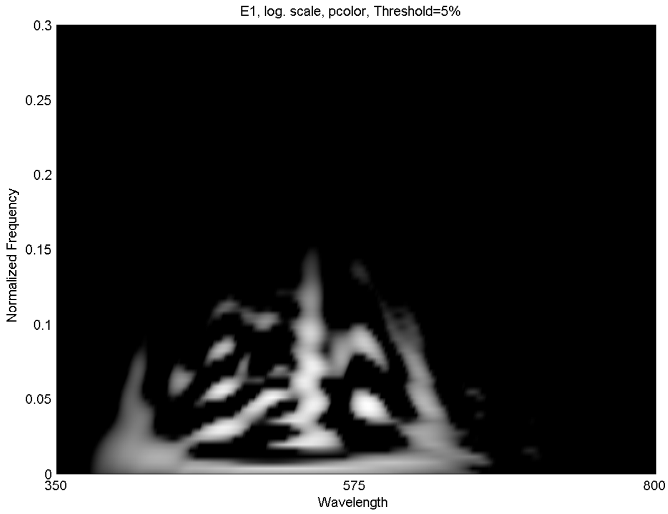

Figure 11: *TFD of PolyluxXL/860 with eleven time-frequency atom*

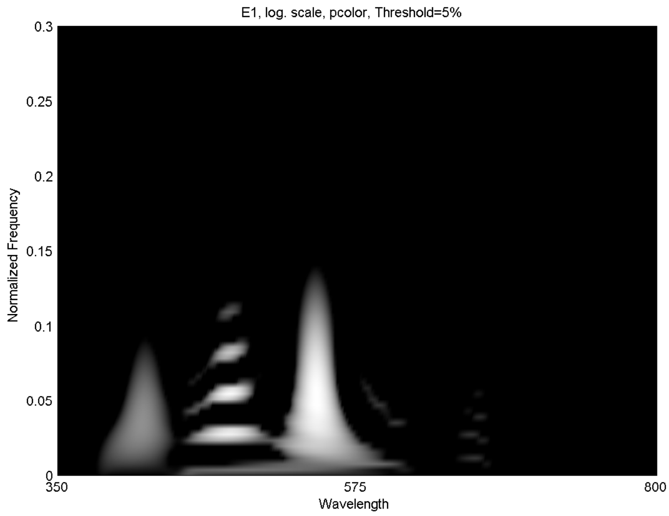

Figure 10: *TFD of PolyluxXL/860 with two time-frequency atoms*

Acknowledgements

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