On Definitions and Construction of Uniform Color Space

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Abstract

A uniform color space has, according to various literature, two definitions: (1) a global uniform color space is a space in which perceptional color difference agrees with the Euclidean distance; (2) a local uniform color space is a space in which discrimination elliptics/ellipsoids are unit circles/spheres everywhere. Unfortunately, it seemed that the relationship between them was not well understood and uniform color spaces have been constructed following these two different definitions independently. In this paper, we discuss the issue from a point of view of global Riemannian geometry and show that these two uniform color spaces are actually equivalent.

Giving perceptive metric in a color space, an efficient algorithm is shown to construct a "pollar coordinate system" for the color space, which is the image of the pollar coordinate system in its uniform space.

Introduction

In 1940's, Wright and MacAdam discovered by psychological experiments that sensitivity of human eyes to small color differences, measured by the jnd (just-noticeable difference) thresholds or discrimination elliptics, is not uniform in the color space but varies drastically [13] [3]. In fact, these discoveries provided definite evidents that a color space is a non-Euclidean or a Riemannian space which has complicated geometric properties. These results have then inspired extensive researches on geometry especially local geometry of the color space [14][2][8].

A uniform color space, or simply a uniform space is, according to various literature, characterized by two features, one global and the other local: (1) It is a space in which the perceptional difference between any pair of colors agrees with the Euclidean distance, or the length of the straightline between the two color vectors; (2) It is a color space where the local curveness is straighten up so that the discrimination elliptics or ellipsoids of color matching at every points are rectified into unit circles or unit spheres centered at these points. In fact, both definitions of the uniform space stated above were adopted by MacAdam himself in his 1971 paper[5].

In fact, the strategies for construction of uniform color spaces simply followed these two above-mentioned definitions of the uniform color space. i.e., either to force the color difference to agree with geometric or Euclidean distance, or force the the jnd or discrimination elliptics in color matching such as MacAdam elliptics into unit circles. In spite of extensive efforts on these two directions, it seems that approaches used in practice mainly based on local and heuristic approximations. As a result, a number of approximative uniform color spaces is available, which used certain complicated nonlinear maps, including several versions of standard uniform spaces recommended by the CIE(Commission International de l'Eclairage) [5] [14][2][8]. However, it seems that there is no method to reach a uniform color space satisfying both global and local definitions.

The reason for this is twofold. The first one is of conceptual, since as shown in this paper the global and local uniform spaces are in fact the same Riemannian space stated from different points of view. In particular, global geometry of a Riemannian space is uniquely determined by its local metric tensor. Therefore, a theoretical treatment is required in order to clear up these confusions.

The second reason is mainly implementational. Rather than starting from global fitting, it is natural and easier to start from local rectification of discrimination ellipsoids. However, in the other direction, to reach a global uniform space using the local metric is a nontrivial task. For instance, one needs global information such as the correspondence between sampling points in the color space and their images in the uniform space, which is notoriously difficult as already known in estimation of the nonlinear maps.

These days uniform color spaces are playing an increasingly important role in transformation between different medias to obtain device-independant or faithful cross-media color reproduction. Furthermore, color perception therefore the Riemann metric of a color space depends on medias, view conditions such as level and color of illuminations and simultaneous contrast effects ect.[7][8]. Although it is possible to obtain formula for colormetric from an ideal setting and under certain assumption, such as invariance with respect to background illumination in [10] or other conditions[14], a general theoretical formulation of uniform color spaces in closed form for all situations are impractical. Moreover, as shown in [11], the color perception for complex images rather than uniform color images should be described by a fibre bundle of color spaces rather than a single space. Therefore, it is desirable to develop a general theory without restricting to specific phenomina and conditions, and fast algorithms to construct, e.g. in real time, uniform color spaces once the information on discrimination threshold or metric of the color space is available.

In this paper, we try to rigorously formulate the issue and discuss pertaining problems using tools of global Riemannian geometry. A duo- space model is used to avoid the threshold problem which prevents proper usage of Riemannian geometry. Then we show equivalence of the local and global definitions of uniform color spaces. Finally, we propose a method based on geodesics of the Riemannian space to obtain a uniform color space. Computer simulation is shown for construction of the uniform space using discriminant elliptics data obtained by MacAdam in 1942[3].

Geometry of color spaces

The color spaces have been known to have nontrivial or non-Euclidean intrinsic geometry in both global and local scales. In the global scale, the disagreement between the perceptional color difference of two color stimuli and the Euclidean distance between the two color vectors in a color space has been a major problem in color matching. Researches have been reported on efforts to construct a global uniform color space, including building the Munsell color space by psychological experiments [14][2]. However, global construction of such a uniform space in a systematical or computational way turned out very nontrivial.

Quantitative study of local geometry of a color space began with the discovery of threshold phenomenon in color matching by Wright, MacAdam and Stiles. Specifically, they found the just noticeable difference (jnd) or just perceptible difference (jpd) thresholds near a center color are not constant but change distinguishablly depending on the direction that the test color deviates from the center color[3]. These discrimination ellipsoids also agree with the standard deviation ellipsoids of the Gaussian distribution of the error probability in color matching.

These results strongly suggest that the color space is a Riemannian space rather than an Euclidean space.

A Riemannian space is a space M with a positive-definite symmetric matrix $G(\boldsymbol{x})$ smoothly defined on $\boldsymbol{x} \in M$ such that the infinitesimal distance near $\boldsymbol{x}$ is measured by $(d\boldsymbol{x}, d\boldsymbol{x})_G = d\boldsymbol{x}^T G(\boldsymbol{x}) d\boldsymbol{x}$. Let $\boldsymbol{x} = (x^1, ..., x^n)^T$, $G = [g_{ij}]$, then

$$(d\boldsymbol{x}, d\boldsymbol{x})_G = g_{ij} dx^i dx^j$$

(Here the Einstein symbol $a^i b_i = \sum_i a^i b_i$ is used).

The matrix G is called a Riemannian metric. The threshold elliptics or ellipsoids induce Riemannian metric in 2D or 3D color spaces. Hereafter we will consider an nD Euclidean space $\mathbb{R}^n$ where the metric G is smoothly defined on points in $\mathbb{R}^n$, and denote the Riemannian space as a pair $M = (\mathbb{R}^n, G)$. Obviously an Euclidean space has the trivial metric $G = I$.

Duo-space model of color spaces

Unfortunately, we still have a problem before one can call the color space a Riemannian space, which seemed has been ignored in most treatments of the issue. In fact, due to the threshold nonlinearility of a color space, the metric is not smoothly defined, which means a color space is not exactly a Riemannian space.

This seems an academic however a serious obstacle in order to use properly Riemannian geometry in the color space. To overcome it, we define a model called a duo-space model $\{C, M\}$, which consists of a underlying color space C with a colormetric G and the thresholds, and a Riemannian space M as a model of the color space which has the same colormetric G as of C but without the thresholds. Since the threshold phenomena only occur within small neighborhoods of the center colors and the space outside of these neighborhoods behaves like as a normal Riemannian space, as long as the global geometry is concerned, one can simply treat the color space C as the Riemannian space M without thresholds. This model makes it possible to use powerful Riemannian geometric techniques in study of the color space, and come back to the thresholded color space if local consideration is necessary.

Thus, hereafter we regard a color space as the Riemannian space in our duo-space model, $M = (\mathbb{R}^3, G)$, where the $\mathbb{R}^3$ is the space of tristimuli, the metric $G(\boldsymbol{x})$ is smoothly defined at every color vector $\boldsymbol{x} \in \mathbb{R}^3$, e.g. induced by the discrimination ellipsoids.

Global distance in color space

Suppose that one had found a map from a uniform color space U to the color space M,

$$\begin{aligned} f : U &\longrightarrow M \\ \boldsymbol{y} &\longmapsto \boldsymbol{x} \end{aligned}$$

As before, we suppose this map is a global diffeomorphism or its Jacobian matrix Df is full rank everywhere. Thus its inverse

$$\begin{aligned} h := f^{-1} : M &\longrightarrow U \\ \boldsymbol{x} &\longmapsto \boldsymbol{y} \end{aligned}$$

is also a global diffeomorphism.

Suppose two color vectors $\boldsymbol{x}, \boldsymbol{x}' \in M$ are mapped to $\boldsymbol{y} := h(\boldsymbol{x}), \boldsymbol{y}' := h(\boldsymbol{x}')$ in the uniform color space U. Then the distance between these two colors is naturally the length of the straight line connecting images of these two color vectors $\overline{\boldsymbol{y}\boldsymbol{y}'}$ in U. The global distance of any two colors $\boldsymbol{x}, \boldsymbol{x}'$ measured in the color space can be defined as the length of the inverse image of the straight line $\overline{\boldsymbol{y}\boldsymbol{y}'}$ under h. For the distance to be well defined, h has to be distance reserving. i.e.

$$d_M(\boldsymbol{x}, \boldsymbol{x}') := \|f(\overline{\boldsymbol{y}\boldsymbol{y}'})\|_M = \|\overline{\boldsymbol{y}\boldsymbol{y}'}\|_U =: d_U(\boldsymbol{y}, \boldsymbol{y}')$$

This kind of inverse images of straight lines in an Euclidean space, or the "straight lines" in a Riemannian space is known as a special class of curves called geodesics. If a spatial curve $\boldsymbol{x}(t) = (x^1, \cdots, x^n)^T$ in a Riemannian space $M = (\mathbb{R}^n, G)$ is smooth for $a \leq t \leq b, \boldsymbol{x}(a) = \boldsymbol{x}, \boldsymbol{x}(b) = \boldsymbol{x}'$, i.e., if $\dot{x}^i$ exist and are continuous, then the integral

$$\theta = \int_a^b \sqrt{g_{ij}\dot{x}^i\dot{x}^j}dt \qquad G = (g_{ij})j \tag{1}$$

exists, and θ is the length between $\boldsymbol{x}(a) = \boldsymbol{x}$ and $\boldsymbol{x}(b) = \boldsymbol{x}'$.

Since the Gamut is metrically complete, it is also geodesically complete. Therefore, between any two points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in M, there is a unique geodesic such that the length of the geodesic is minimal.

Define the Christoffel symbols Γ^i_{jk} as

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}) \tag{2}$$

where $G^{-1} = (g^{ij})$, the geodesics of the Levi-Civita connection are defined by the second order ODE:

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0. \tag{3}$$

Thus the global distance between any two colors $\boldsymbol{x}$ and $\boldsymbol{x}'$ is equivalently defined as the geodesic distance or the arc length of the geodesic $\boldsymbol{x}(t)$ connecting $\boldsymbol{x}, \boldsymbol{x}'$

$$d(\boldsymbol{x}, \boldsymbol{x}') := \int_a^b \sqrt{g_{ij}\dot{x}^i\dot{x}^j}dt \tag{4}$$

such that $\boldsymbol{x}(a) = \boldsymbol{x}, \boldsymbol{x}(b) = \boldsymbol{x}'$.

Definitions of local and global uniform spaces

Let $M = (\mathbb{R}^n, G(\boldsymbol{x})), n = 2, 3$ be a color space as a Riemanninan space. Suppose a map from another Riemannian space $U = (\mathbb{R}^n, H(\boldsymbol{y}))$, i.e., a uniform space of M to M itself.

$$\begin{aligned} f \,:\, U &\longrightarrow M \\ \boldsymbol{y} &\longmapsto \boldsymbol{x} \end{aligned}$$

In fact, a uniform color space is an Euclidean space $(\mathbb{R}^n, I)$.

Definition 1. *(Local uniform space 1)*

A Riemannian space $U_1 = (\mathbb{R}^n, I)$ is a locally uniform color space of the color space $M = (\mathbb{R}^n, G)$ if there is an isometry (bijective local isometry) f_1

$$\exists\, f_1 \,:\, U_1 \longrightarrow M$$

s.t. $(f_1)_*(I) = G$, i.e.

$$(d\boldsymbol{y}, d\boldsymbol{y}')_I = (df_1(\boldsymbol{y}), df_1(\boldsymbol{y}'))_G \;\; or \;\; \|d\boldsymbol{y}\|_I = \|df_1(\boldsymbol{y})\|_G$$

Definition 2. *(Local uniform space 2)*

A Riemanninan space $U_2 = (\mathbb{R}^n, I)$ is a locally uniform color space of $M = (\mathbb{R}^n, G)$ if there is a smooth map f_2

$$\exists\, f_2 \,:\, U_2 \longrightarrow M$$

s.t.

$$H(\boldsymbol{y}) = Df_2(\boldsymbol{y})^T G(\boldsymbol{x}) Df_2(\boldsymbol{y}) = I \tag{5}$$

Therefore a local uniform space is a color space where the local curveness is straighten up. In particular, the metric G in M is transformed by f_2^{-1} to the identity matrix $H(\boldsymbol{y}) = I$, or locally the discrimination elliptics or ellipsoids in M are rectified into unit circles or unit spheres centered at $\boldsymbol{y}$ in U.

Lemma 1. *The two definitions of local uniform space: Definition 1 and definition 2 are equivalent.*

Proof: Assume f_1 exists. Since $\forall d\boldsymbol{y}, d\boldsymbol{y}' \in T_{\boldsymbol{y}}U, \boldsymbol{x} = f_1(\boldsymbol{y}), d\boldsymbol{x} = Df_1(\boldsymbol{y})d\boldsymbol{y}, d\boldsymbol{x}' = Df_1(\boldsymbol{y})d\boldsymbol{y}'$,

$$\begin{aligned} & (d\boldsymbol{y}, d\boldsymbol{y}')_I = (d\boldsymbol{x}, d\boldsymbol{x}')_G \\ \Longleftrightarrow \quad & d\boldsymbol{y}^T d\boldsymbol{y}' = d\boldsymbol{y}^T Df_1(\boldsymbol{y})^T G(\boldsymbol{x}) Df_1(\boldsymbol{y}) d\boldsymbol{y}' \\ \Longleftrightarrow \quad & Df_1(\boldsymbol{y})^T G(\boldsymbol{x}) Df_1(\boldsymbol{x}) = I \end{aligned}$$

Thus one can chose $f_2 = f_1, U_2 = U_1$. The other direction is similar. □

Definition 3. *(Global uniform space 1.)*

A Riemannian space $U_3 = (\mathbb{R}^n, I)$ is a globally uniform color space of $M = (\mathbb{R}^n, G)$ if there is a smooth map f_3,

$$\exists\, f_3 \,:\, U_3 \longrightarrow M$$

which preserves lengths. i.e. The length of the arc $f_3(\overline{\boldsymbol{y}\boldsymbol{y}'})$ equals to $\sqrt{(\boldsymbol{y}-\boldsymbol{y}')^T(\boldsymbol{y}-\boldsymbol{y}')}$

$$\int_{f_3(\boldsymbol{y})}^{f_3(\boldsymbol{y}')} \sqrt{g_{ij}\dot{x}^i\dot{x}^j}ds = \sqrt{(\boldsymbol{y}-\boldsymbol{y}')^T(\boldsymbol{y}-\boldsymbol{y}')}$$

along the arc $f_3(\overline{\boldsymbol{y}\boldsymbol{y}'})$.

Definition 4. *(Global uniform space 2.)*

A Riemannian space $U_4 = (\mathbb{R}^n, I)$ is a globally uniform color space of $M = (\mathbb{R}^n, G)$ if there is a smooth map f_4,

$$\exists\, f_4 \,:\, U_4 \longrightarrow M$$

the image $f_4(\overline{\boldsymbol{y}\boldsymbol{y}'})$ of the straightline $\overline{\boldsymbol{y}\boldsymbol{y}'}$ between $\boldsymbol{y}$ to $\boldsymbol{y}'$ in U_4 is a geodesic between $f_4(\boldsymbol{y})$ and $f_4(\boldsymbol{y}')$ in M and f_4 preserves global distance. i.e.

$$\begin{aligned} \forall \boldsymbol{y}, \boldsymbol{y}' \in U_4, \qquad & f_4(\boldsymbol{y}), f_4(\boldsymbol{y}') \in M \\ d_M(f_4(\boldsymbol{y}), f_4(\boldsymbol{y}')) = & \sqrt{(\boldsymbol{y}-\boldsymbol{y}')^T(\boldsymbol{y}-\boldsymbol{y}')}. \end{aligned} \tag{6}$$

where

$$d_M(f_4(\boldsymbol{y}), f_4(\boldsymbol{y}')) := \int_a^b \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, ds \tag{7}$$

is the integral along the geodesic $\boldsymbol{x}(s) : \boldsymbol{x}(a) = f_4(\boldsymbol{y})$, $\boldsymbol{x}(b) = f_4(\boldsymbol{y}')$. i.e.

$$\int_a^b \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, ds = \sqrt{(\boldsymbol{y}-\boldsymbol{y}')^T(\boldsymbol{y}-\boldsymbol{y}')} \tag{8}$$

Thus, in a globally uniform color space, the perceptional difference between any pair of colors, or the geodesic distance (i.e. the line integral along the geodesic) between the two colors $f_4(\boldsymbol{y})$ and $f_4(\boldsymbol{y}')$ agrees with the Euclidean distance, or the length of the straightline between the two color vectors.

Lemma 2. *The two definitions of global uniform space: Definition 3 and definition 4 are equivalent.*

Proof: First, Def. 3 $\Longleftarrow$ Def. 4 is trivial;

For Def. 3 $\Longrightarrow$ Def. 4, assume f_3 is length preserving, i.e. the length of the image of straightline $\overline{\boldsymbol{y}\boldsymbol{y}'}$ between $f(\boldsymbol{y})$ and $f(\boldsymbol{y}')$ equals $\sqrt{(\boldsymbol{y}-\boldsymbol{y}')^T(\boldsymbol{y}-\boldsymbol{y}')}$. Since this length in the Euclidean space is minimal, the length of the curve $f_3(\overline{\boldsymbol{y}\boldsymbol{y}'})$ is also minimal. Thus, the image $f(\overline{\boldsymbol{y}\boldsymbol{y}'})$ must be a geodesic according to the uniqueness of geodesics. Of course the map f_3 preserves geodesic distance. i.e. one can assume $f_4 = f_3, U_4 = U_3$. □

Local and global uniform spaces are equivalent

Theorem 1. *The definitions of global uniform space and local uniform space are equivalent.*

Proof:
Def.1 $\Longleftrightarrow$ Def. 2 : By Lemma 1, one has $f_1 = f_2, U_1 = U_2$;
Def. 2 $\Rightarrow$ Def. 3: If $f_2 : U_2 \longrightarrow M$ exists. It can be locally linearized as the Jacobian matrix map $D_2(\boldsymbol{y}) = Df_2 : T_{\boldsymbol{y}}U_2 \longrightarrow T_{\boldsymbol{x}}M$, we see $d\boldsymbol{x} = D_2 d\boldsymbol{y}$ s.t. $D_2^T G D_2 = I$. Denote the straightline $\overline{\boldsymbol{y}\boldsymbol{y}'}$ as $\boldsymbol{y}(s), a \le s \le b, \boldsymbol{y}(a) = \boldsymbol{y}, \boldsymbol{y}(b) = \boldsymbol{y}'$. Then along the image of $\overline{\boldsymbol{y}\boldsymbol{y}'}$: $f_l(\overline{\boldsymbol{y}\boldsymbol{y}'}), \forall d\boldsymbol{y} \in T_{\boldsymbol{y}}U_2$,

$$\begin{aligned} d\boldsymbol{y}^T d\boldsymbol{y} = d\boldsymbol{x}^T G d\boldsymbol{x} &= d\boldsymbol{y}^T D_2^T G D_2 d\boldsymbol{y} \\ \int_{f_l(\boldsymbol{y})}^{f_l(\boldsymbol{y}')} \sqrt{d\boldsymbol{x}^T G d\boldsymbol{x}} &= \int_a^b \sqrt{\dot{\boldsymbol{x}}^T G \dot{\boldsymbol{x}}} ds \\ = \int_a^b \sqrt{\dot{\boldsymbol{y}}^T D_2^T G D_2 \dot{\boldsymbol{y}}} ds &= \int_a^b \sqrt{\dot{\boldsymbol{y}}^T \dot{\boldsymbol{y}}} ds \\ = \int_{\boldsymbol{y}}^{\boldsymbol{y}'} \sqrt{d\boldsymbol{y}^T d\boldsymbol{y}} &= \sqrt{(\boldsymbol{y}-\boldsymbol{y}')^T(\boldsymbol{y}-\boldsymbol{y}')} \end{aligned}$$

Thus one can assume $f_3 = f_2, U_3 = U_2$ in Def. 3;
Def. 3 $\Longleftrightarrow$ Def. 4: By lemma 2;
Def. 4 $\Rightarrow$ Def. 2: Fix a $\forall \boldsymbol{y} \in U_4, \boldsymbol{x} = f_4(\boldsymbol{y})$. Choose $\forall d\boldsymbol{y} \in T_{\boldsymbol{y}}U_4$. The straightlines from $\boldsymbol{y}$ along the direction of $d\boldsymbol{y}$, say to certain $\boldsymbol{y}'$, are mapped by f_4 to the geodesics in M from $\boldsymbol{x}$ to $\boldsymbol{x}' = f_4(\boldsymbol{y}')$. Then

$$\begin{aligned} \sqrt{(\boldsymbol{y}-\boldsymbol{y}')^T(\boldsymbol{y}-\boldsymbol{y}')} &= \int_{\boldsymbol{y}}^{\boldsymbol{y}'} \sqrt{d\boldsymbol{y}^T d\boldsymbol{y}} \\ = \int_{f_4(\boldsymbol{y})}^{f_4(\boldsymbol{y})} \sqrt{d\boldsymbol{x}^T G d\boldsymbol{x}} &= \int_{\boldsymbol{y}}^{\boldsymbol{y}'} \sqrt{d\boldsymbol{y}^T D_4^T G D_4 d\boldsymbol{y}} \\ \Longrightarrow \quad D_4^T G D_4 &= I \quad \forall \boldsymbol{x} \in M \end{aligned}$$

where $D_4 := Df_4$. Thus one can assume $f_2 := f_4, U_2 = U_4$. □

Difficulty in construction of a uniform space

Existing approaches to construct uniform color space can be roughly divided into two categories, each follows either the global or the local definition of uniform spaces. According to the conclusion of the last chapter, one should be able to reach the same uniform color space following either of the global or the local definitions. However, accessibility of the approaches based on different definitions are very different in practice.

For approaches following the global definition of the uniform color space, the Munsell color space e.g. can be regarded as a model, although it has merely approximative agreement between the perceptional color difference and the Euclidean distance of color vectors. However, since it is obtained by extensive psychological experiments, transforms between it and the other color spaces seems very difficult.

To obtain such a transform, which should be computational rather than psychological experiments based, most researches assumed a nonlinear map from the color space to its uniform space. This global uniformization map is approximated by rational functions, estimated from the correspondence between sampling points in the color space and their images in the uniform space. The correspondence however are generally unknown in priori. Hence the images are estimated from empirical color difference formula. Variations have been proposed to improve the accuracy of such maps but their expressions become increasingly complicated [14] [2][8]. e.g., the standard version CIELAB space recommended by the CIE behaves better for global color difference comparing with the CIELUV space, but can not locally uniformize the MacAdam elliptics [14][8].

In fact, it is known in Riemannian geometry that there is no general way to determine local geometry of a space assuming certain global properties. On the other hand, global geometrical properties of a Riemannian space are uniquely determined by its local Riemannian metric tensor. Therefore it would be more natural to start with the local uniformization.

MacAdam adopted both the global and local definition of the uniform color space. However, instead of using the local metric to obtain geodesics, he built his uniform color space by again a global nonlinear map whose images are empirically estimated from a modified color difference formula of Frierel. As a result he tested the MacAdam elliptics in this uniform space were approximated rectified near to unit circles[5]. Along this direction, the standard CIELUV space shows better local uniformation to rectify the MacAdam elliptics than the CIELAB space but worse in global uniformization [14] [2][8].

In fact, to construct an exact uniform space, one could use global Riemannian geometry based on the local metric tensor. We will use mainly the geodesics for such combination of local and global information. Another possible approach of global uniformation is to build a grid in the 3D color space, as the inverse image of the orthogonal coordinate grid in an Euclidean space.(e.g. [12]) Since the scale of the uniform space or the inverse map of the uniformization map has already been imprinted in the grids, this kind uniform space is very convenient in practice than the nonlinear map method, where the inverse map is hard to calculate. In fact, a complete uniformization of a 3D Riemannian space is equivalent to build a 3D grid in the space consisting of orthogonal/parallel geodesics. This is the approach we used in this paper.

Algorithm to construct uniform space

Bellow, we show an algorithm to construct a “pollar coordinate system” for the color space, which is the image of the pollar coordinate system in an Euclidean space or the uniform space.

First we need some notations. For a tangent vector at $\boldsymbol{x} \in M$, $\boldsymbol{a} \in T_{\boldsymbol{x}}M$, denote the geodesic starting from $\boldsymbol{x}$ with the initial vector $\boldsymbol{a}$ as $\gamma(t, \boldsymbol{a})$, then an exponential map is defined as

$$\exp_{\boldsymbol{x}}(\boldsymbol{a}) = \gamma(1, \boldsymbol{a})$$

A normal neighborhood V at a point $\boldsymbol{x} \in M$ is the image of the exponential map of some neighborhood $U \subset T_{\boldsymbol{x}}M$. The coordinate of the normal neighborhood is called Riemannian or normal coordinate, which can be regarded as an extension of pollar coordinate. We assume the color space M can be defined as a normal neighborhood of a $\boldsymbol{x} \in M$ for certain $U \subset T_{\boldsymbol{x}}M$.

Algorithm

Step 1: For a 2D color space, choose points $\{\boldsymbol{a}_k, k = 1, ..., K\}$ which are uniformly distributed points on the threshold elliptics at $\boldsymbol{x} \in M$, with angles $\{k\phi_0, \phi_0 = 2\pi/K$ for prechosen resolutions K ;

In 3D case, choose points $\{\boldsymbol{a}_{ij}, i = 1, ..., K_1, j = 1, ..., K_2\}$ which form a uniformly distributed lattice on the threshold ellipsoid at $\boldsymbol{x} \in M$, with angles $(i\phi_0, j\psi_0)$, $\phi_0 = 2\pi/K_1$, $\psi_0 = 2\pi/K_2$ for prechosen resolutions K_1, K_2;

Step 2: Draw geodesics $\{\gamma_{\boldsymbol{p}}(\theta, \boldsymbol{a}_k), k = 1, ..., K\}$ in the 2D case and geodesics $\gamma_{\boldsymbol{x}}(\theta, \boldsymbol{a}_{ij}), i = 1, ..., K_1, j = 1, ..., K_2$ in the 3D case;

Step 3: Draw the closed curve in the 2D case and closed surface in the 3D case by connecting the points on all geodesics $\{\gamma_{\boldsymbol{x}}(\theta, \boldsymbol{a}_k), k = 1, ..., K\}$ or $\gamma_{\boldsymbol{x}}(n\theta_0, \boldsymbol{a}_{ij}), i = 1, ..., K_1, j = 1, ..., K_2$;

Step 4: Output the data $\{(n, k), n = 1, ..., N, k = 1, ..., K\}$ or $\{(n, i, j), n = 1, ..., N, i = 1, ..., K_1, j = 1, ..., K_2\}$ as the 2D or 3D normal coordinates.

Simulation

A tentative simulation is carried out using the discrimination elliptics data obtained in [3]. The long and short axises of the elliptics are interpolated using Akima's algorithm [1] with interval 0.01. These data are then smoothed using a Gaussian filter with variant 0.06 on a 25×25 neighborhood. The geodesics are obtained using the third order Runge- Kutta method with resolution 0.001, started from the CIE standard white $D65 = (0.313, 0.329)$.

The first order partial derivatives in Christoffel symbol are calculated simply using the central difference. The resulting geodesic grid is shown in Figure 1.

Conclusion

In this paper, we formulated and discussed the problem of construction of uniform color space from the viewpoint of global Riemannian geometry. It is shown that both definitions are actually equivalent. An algorithm for construction of a uniform color space was presented. Future works including comparison of this algorithm with the algorithms building orthogonal grid in the color space[12].

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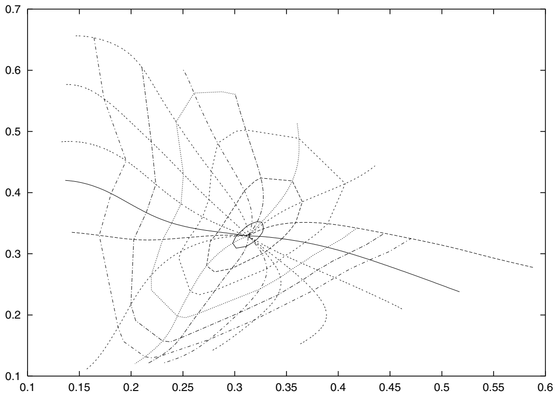

Figure 1: *A geodesic grid*

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Biography

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