Vector Correlation of Color Images

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Abstract

Auto-correlation and cross-correlation have been used in signal and image processing for many years, but have only recently been applied to color images. Since the correlation of individual signals and gray-scale images yields a measure of structural similarity, in each of their dimensions, the correlation of *n*-dimensional signals should yield similarity measures of *n*-dimensional space. This paper presents, for the first time, evidence that a vector correlation peak encodes both the structural similarity and also a mapping of the vector-space rotation between two color images.

Introduction

The basic use of correlation is to determine how similar, or dissimilar, one signal is from another. This knowledge of similarity has a multitude of uses in signal and image processing. Two of the more popular are filtering and registration. For one- and two-dimensional signals the measure of similarity is given by the magnitude of the response from a correlation operation. This measure has been of considerable use but encodes nothing about the transformation from one signal into the other. As part of our investigation into use of correlation techniques on color images, we have discovered that the extra information contained within the hypercomplex (quaternion) representation of color encodes both the similarity of structure and the vector-space mapping between two images. Although our work demonstrates correlation of RGB images, the theory applies equally to other vector images.

Due to the novelty of the theory presented in this paper, many of the practical applications of this discovery have yet to be realized. However, an immediate application could be the removal of global color distortions. Such distortions are evident in color images captured at widely differing time periods. This could be over a period of hours, the difference between the sunlight illuminating a scene in the morning and in the afternoon, or over a number of years, where film type and/or degradation may be apparent. Our recent paper [1] demonstrated color image registration, using single-stage vector phase correlation, in spite of noise and color-space distortions. This paper offers the alternative perspective; image processing solutions which are tuned to, or depend on, the color content.

This paper begins with two sections giving a short review of hypercomplex processing of color images. This is followed by a section which identifies our previous work on correlation, adds new understanding of the early results and introduces the theory and proof of the encoding of vector-space rotations. The penultimate section describes our latest experiments and demonstrates the use of the new theory and understanding of hypercomplex correlation.

Hypercomplex Numbers and Color Images

Quaternions (also referred to as *hypercomplex numbers*) are an extension of complex numbers to four dimensions. Originally proposed by Hamilton in 1843 [2], quaternions have been used to encode and compute transformation in three-dimensional space for many years. They can be considered as a complex number with a vector imaginary part consisting of three mutually orthogonal components. In Cartesian form, a quaternion is usually represented as, $q = w + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, where w, x, y and z are all real and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the complex operators which obey,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \qquad \text{and}$$
$$\boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{jk} = \boldsymbol{i}, \boldsymbol{ki} = \boldsymbol{j}, \boldsymbol{ji} = -\boldsymbol{k}, \boldsymbol{kj} = -\boldsymbol{i}, \boldsymbol{ik} = -\boldsymbol{j}$$

Given a quaternion $q = w + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ its quaternion conjugate is $\overline{q} = w - x\,\boldsymbol{i} - y\,\boldsymbol{j} - z\,\boldsymbol{k}$ and its modulus given by,

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2}$$

A *pure* quaternion has a zero real part ($w = 0$) and a *unit* quaternion has a unit modulus. It is often useful to consider a quaternion as composed of a Scalar and a Vector part, represented by $q = S(q) + \boldsymbol{V}(q)$, where $\boldsymbol{V}(q) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$.

An RGB color image may be represented using quaternions by encoding the red, green and blue channels of the image as a pure quaternion such that the image function is given by,

$$f(x, y) = r(x, y)\,\boldsymbol{i} + g(x, y)\,\boldsymbol{j} + b(x, y)\,\boldsymbol{k}$$

where $r(x, y)$ is the red component and similar from the green and blue. This representation effectively equates the

RGB color cube to the right-hand coordinate frame imposed by the imaginary part of quaternion-space. However, this is not an exclusive representation. An alternative could be used to define the origin of the coordinate frame to be coincident with the center of the RGB color cube, or any other position, in any other color-space with appropriate scaling.

Polar Form and Visualization

In order to simplify both the notation and explain rotations in color-space it is easier to consider the polar form of a quaternion. Euler's formula for the complex exponential generalizes to the hypercomplex form:

$$e^{\boldsymbol{\mu}\Phi} = \cos\Phi + \boldsymbol{\mu}\sin\Phi \tag{1}$$

where $\boldsymbol{\mu}$ is a pure quaternion. Any quaternion may be represented in polar form by,

$$q = |q|\ e^{\boldsymbol{\mu}\Phi}$$

where $\boldsymbol{\mu}$ and Φ are referred to as the *eigenaxis* and the *eigenangle* respectively, We generally refer to the former simply as the *axis* and the latter as the *phase*. The eigenaxis, or axis, is computed as $\boldsymbol{\mu} = \boldsymbol{V}(q)/\,|\boldsymbol{V}(q)|$ with the only exception being when $\boldsymbol{V}(q) = 0$, in which case $\boldsymbol{\mu}$ is undefined. The eigenangle, or phase, is computed as

$$\Phi = \tan^{-1}\frac{|\boldsymbol{V}(q)|}{S(q)}$$

and is always positive in the range $0 \leq \Phi \leq \pi$. If the quaternion is zero the phase is undefined.

The components of the polar form of a quaternion may be visualized as follows. The modulus can be represented by a gray-scale image, appropriately scaled with logarithmic scaling such that:

$$l(q) = \log(1 + |q|)/\log(1 + K)$$

where K is the largest modulus in the image. The phase can be visualized by a color represented based on the IHS color-space [3]. By using an appropriate transformation and clipping values outside the valid range we represent the phase by half of the hue range. This effectively translates a phase of zero to the reference hue of red with a phase of $\pi/2$ represented by a green hue and a phase of π by a cyan hue. The use of only half the hue range is deliberate since the phase is only unique in the range 0 to π. A phase outside this range is negated by the use of the opposite axis which is more mathematically tractable.

Due to the orthogonal nature of its coordinate frame and the fact that it is defined by a pure quaternion, with three non-zero components, the axis is readily represented by an RGB color cube. However the axis can take up any arbitrary direction in 3-space so it should be considered as being constrained by a unit sphere. The sphere can then be imagined as occupying the volume inside the color cube. The representation is scaled such that an axis in the direction of either of the six faces is represented by the fully saturated value of the appropriate color. This representation is convenient for human interpretation, so the fact that neither the saturated values equating to the vertices of the color cube can be used, due to the constraint of the unit sphere, is of little importance compared to the structural information that is conveyed. Although both the eigenaxis and eigenangle have degenerate cases where they are undefined, these can be easily visualized using the representations given above. An undefined phase is represented by the color black and an undefined axis is represented by mid-gray to indicate the center of the RGB color cube.

Quaternions provide a convenient representation for rotations in 3-space. A rotation through an angle α about an axis $\boldsymbol{\mu}$ is represented by the quaternion operator $R[\quad]\bar{R}$, where $R = e^{\boldsymbol{\mu}\frac{\alpha}{2}}$ and the square brackets indicate space for the quaternion which is to be operated upon. The use of the rotation operator has been demonstrated in [4], where a number of quaternion filters, based on traditional gray-scale filters, were derived using the standard gray-scale convolution mask technique.

Vector Correlation

In [5] the cross-correlation of two images was extended to hypercomplex images using quaternion arithmetic:

$$r(m,n) = \sum_{q=0}^{M-1}\sum_{p=0}^{N-1} f(q,p)\overline{g(q-m,p-n)} \tag{2}$$

where the shift operation on $g(m,n)$ was implemented cyclically using modulo arithmetic. If the images $f(m,n)$ and $g(m,n)$ are the same the auto-correlation of the image is computed. If the mean, or DC level, of each image is subtracted first the cross-covariance is obtained. Direct evaluation of the cross-correlation function is impractical for all but the smallest images due to the high computational cost, $O(N^4)$ for a $N \times N$ image. This necessitates the use of fast Fourier transforms, the hypercomplex form of which was first published in [6].

$$F(v,u) = \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-\mu 2\pi\left(\frac{mv}{M}+\frac{nu}{N}\right)} \tag{3}$$

with the reverse transform:

$$f(m,n) = \frac{1}{\sqrt{MN}}\sum_{v=0}^{M-1}\sum_{u=0}^{N-1} F(v,u)\, e^{\mu 2\pi\left(\frac{mv}{M}+\frac{nu}{N}\right)} \tag{4}$$

In this transform pair, $\boldsymbol{\mu}$ is an arbitrary unit vector, but the standard, general case, choice for RGB images is $\boldsymbol{\mu} = (i + j + k)/\sqrt{3}$ which is aligned to the luminance, or *grayline*, axis of the color cube. The form of the transforms given in Equations (3) and (4) is not unique but one of a pair of different, but closely related, transforms which the authors call *transpose transforms*. The transpose transform is obtained by interchanging the hypercomplex exponential from the right with the function from the left. The ordering of these terms is important due to the non-commutative nature of quaternions and such exchanges lead to different results. The application of these transforms therefore requires particular care. In this paper the transform in Equation (3) is denoted by $\mathcal{F}^R$, its reverse in Equation (4) by $\mathcal{F}^{-R}$ and the related transpose transform, with the hypercomplex exponential on the left, by $\mathcal{F}^L$ and its reverse by $\mathcal{F}^{-L}$.

Computing the cross power spectrum of two hypercomplex Fourier transformed images requires more than the standard $F(v,u)\overline{G(v,u)}$. It is therefore necessary to consider the decomposition of a quaternion into its parallel and perpendicular components. Given two pure quaternions $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{u}$ may be *decomposed* into components *parallel* and *perpendicular* to $\boldsymbol{v}$ such that:

$$\boldsymbol{u}_\perp = \tfrac{1}{2}(\boldsymbol{u} + \boldsymbol{v}\boldsymbol{u}\boldsymbol{v}), \quad \boldsymbol{v} \perp \boldsymbol{u}_\perp \tag{5}$$

$$\boldsymbol{u}_\parallel = \tfrac{1}{2}(\boldsymbol{u} + \boldsymbol{v}\boldsymbol{u}\boldsymbol{v}), \quad \boldsymbol{v} \parallel \boldsymbol{u}_\parallel \tag{6}$$

This extends to full quaternions such that the decomposition of q about some vector $\boldsymbol{v}$ yields $q_\parallel = S(q) + \boldsymbol{V}_\parallel(q)$ and $q_\perp = \boldsymbol{V}_\perp(q)$. The quaternion $\boldsymbol{u}$ can be recovered by adding the two components back together. The proof for these is given in [7] and can be applied in both the spatial and spatial-frequency domains.

Parallel quaternions (strictly co-planar quaternions, or quaternions with parallel vector parts) commute. If q is a full quaternion and $\boldsymbol{p}$ is a pure quaternion, or a vector, where $q \perp \boldsymbol{p}$, they can be reordered such that $q\boldsymbol{p} = \boldsymbol{p}\overline{q}$. The proof for this is given by:

$$q\boldsymbol{p} = [S(q) + \boldsymbol{V}(q)]\boldsymbol{p} = S(q)\boldsymbol{p} + \boldsymbol{V}(q)\boldsymbol{p} \quad \text{and}$$
$$\boldsymbol{p}\overline{q} = \boldsymbol{p}[S(q) - \boldsymbol{V}(q)] = S(q)\boldsymbol{p} - \boldsymbol{p}\boldsymbol{V}(q) \tag{7}$$

Two perpendicular vectors (here $\boldsymbol{p}$ and $\boldsymbol{V}(q)$) reverse their sign on reordering since the product of any two vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ is given by $\boldsymbol{v}\boldsymbol{u} = -\boldsymbol{v}\cdot\boldsymbol{u} + \boldsymbol{v}\times\boldsymbol{u}$, and when $\boldsymbol{v} \perp \boldsymbol{u}$ the dot product is zero.

Using the transforms given in Equations (3) and (4), together with the theory of decompositions, the standard Wiener-Khintchine theorem, which relates the auto-correlation and the power spectral density of a scalar image by $\mathcal{F}\{r(m,n)\} = |F(v,u)|^2$, where $r(m,n)$ is the auto-correlation function and $F(v,u)$ is the Fourier transform of the image $f(m,n)$, was extended in [7] to give a generalized hypercomplex form:

$$r(m,n) = \mathcal{F}^{-R}\left\{F^R[\boldsymbol{u}]\,\overline{G^R_\parallel[\boldsymbol{u}]}\right\} + \mathcal{F}^R\left\{F^R[\boldsymbol{u}]\,\overline{G^R_\perp[\boldsymbol{u}]}\right\} \tag{8}$$

where; $[\boldsymbol{u}] = (v,u)$, $G[\boldsymbol{u}] = \mathcal{F}\{g(m,n)\}$, $G_\parallel[\boldsymbol{u}] \parallel \boldsymbol{\mu}$ and $G_\perp[\boldsymbol{u}] \perp \boldsymbol{\mu}$. Note that all of these are *right-hand* transforms.

Recently in [1], an alternative form, with mixed left and right transforms, was derived:

$$\begin{aligned} R(m,n) &= \overline{F^L[\boldsymbol{u}]}G^R_\parallel[\boldsymbol{u}] + \overline{F^{-L}[\boldsymbol{u}]}G^R_\perp[\boldsymbol{u}] \\ r(m,n) &= \mathcal{F}^{-R}\{R(m,n)\} \end{aligned} \tag{9}$$

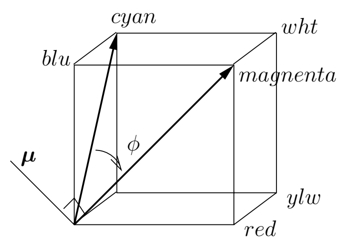
and used to compute the vector phase correlation given by:

$$p(m,n) = \mathcal{F}^{-R}\left\{\frac{R(m,n)}{|R(m,n)|}\right\} \tag{10}$$

The advantage of this form is that it has a single reverse transform to return to the spatial domain, but both forms have the same computational expense. Since there is at least a two-fold symmetry in quaternion space, there are, in general, equivalent forms for any given hypercomplex operation.

Interpretation

In [5] initial results proved the concept of hypercomplex correlation and gave an insight into the encoding of more information than just the structural similarity between two images. These results were, however, confused by the effective DC level present in the images. In order to make sense of the correlation and covariance results, the effect of the DC level must be taken into account. We have therefore returned to the most pathological of cases, that of plain color images. These have no structural content and are effectively DC signals. Computing the cross-correlation of different colors yields the obvious maximum modulus but more importantly encodes a perfect mapping, from one color to the other, in the phase and axis responses. Figure 1 shows the relationship of the phase and axis quantities for correlating a plain cyan image with a plain magenta image. The values were derived directly from the correlation response of an experiment and are equivalent to a 60° rotation about an axis of $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$.



phase: $\phi = 1.0472$

axis: $\boldsymbol{\mu} = (-0.5773, -0.5773, 0.5773)$

Figure 1: Phase & axis relationship in the correlation of cyan to magenta.

From Equation (2), consider only a single point, say the central pixel, in the correlation response between the two plain, single-color, images such that:

$$r\left[\frac{m}{2}, \frac{n}{2}\right] = C\,e^{\boldsymbol{\mu}_c \phi} \tag{11}$$

The correlation of this pathological case yields a flat response. Ignoring the size of the image, this can be considered as the product of two quaternions. Any quaternion can be expressed as the product of two pure quaternions. Therefore, the correlation function can be modeled by:

$$C\,e^{\boldsymbol{\mu}_c\phi} = A\,e^{\boldsymbol{\mu}_1\frac{\pi}{2}}\,\overline{B\,e^{\boldsymbol{\mu}_2\frac{\pi}{2}}} = A\,\boldsymbol{\mu}_1\,\overline{B\,\boldsymbol{\mu}_2} \tag{12}$$

where the simplification is given by the generalized Euler's formula from Equation (1) with $\Phi = \pi/2$.

Image B can be considered as an appropriately scaled, color-space rotated version of A, given by a rotation about an axis $\boldsymbol{\mu}$, where $\boldsymbol{\mu} \perp \boldsymbol{\mu}_1 \& \boldsymbol{\mu}_2$, by some "unknown" angle, θ, such that:

$$B\boldsymbol{\mu}_2 = \frac{B}{A}\left(e^{\boldsymbol{\mu}\frac{\theta}{2}}\, A\boldsymbol{\mu}_1\, e^{-\boldsymbol{\mu}\frac{\theta}{2}}\right) \qquad (13)$$

Substituting in Equation (12) yields,

$$C\, e^{\boldsymbol{\mu}_c \phi} = A\, \boldsymbol{\mu}_1 \overline{\frac{B}{A}\, e^{\boldsymbol{\mu}\frac{\theta}{2}}\, A\boldsymbol{\mu}_1\, e^{-\boldsymbol{\mu}\frac{\theta}{2}}} \qquad (14)$$

which, by removing the conjugate and canceling terms, simplifies to:

$$C\, e^{\boldsymbol{\mu}_c \phi} = -AB\, \boldsymbol{\mu}_1\, e^{\boldsymbol{\mu}\frac{\theta}{2}}\, \boldsymbol{\mu}_1\, e^{-\boldsymbol{\mu}\frac{\theta}{2}} \qquad (15)$$

Since $\boldsymbol{\mu} \perp \boldsymbol{\mu}_1$ we can apply the reordering rule from Equation (7) (where $e^{\boldsymbol{\mu}\frac{\theta}{2}}$ is a full quaternion and $\boldsymbol{\mu}_1$ is an axis vector, or pure quaternion), to Equation (15) such that,

$$C\, e^{\boldsymbol{\mu}_c \phi} = -AB\, \boldsymbol{\mu}_1\, \boldsymbol{\mu}_1\, e^{-\boldsymbol{\mu}\frac{\theta}{2}}\, e^{-\boldsymbol{\mu}\frac{\theta}{2}} = AB\, e^{-\boldsymbol{\mu}\theta}$$

where $\boldsymbol{\mu}_1\, \boldsymbol{\mu}_1 = -1$.

The axis of rotation, $\boldsymbol{\mu}$, is therefore given by the axis of the correlation peak, $\boldsymbol{\mu}_c$ and the rotation angle, θ, by the negated phase of the correlation peak, ϕ.

Latest Results

Extending the theory of color-space mapping given above to natural images, Figure 2 shows two versions of the Lena image with the second color shifted by 60° about the red/cyan axis. The color-space rotation is produced by convolving the image with the rotation operator described earlier. Computing the cross-correlation of these images

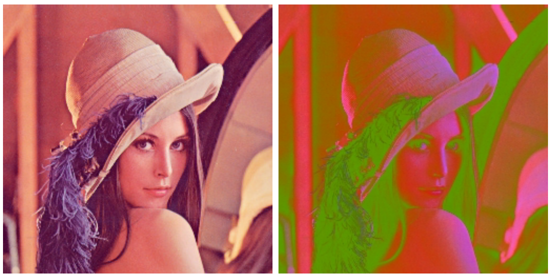


Figure 2: Original and color-space rotated versions of the Lena image.

does not produce any useful information because both the phase and axis response are dominated by the DC content in the images. However, computing the cross-covariance and extracting the phase and axis from the same position as the peak modulus response yields the color-space mapping from the first to the second image. Using this information, a quaternion rotation operator can be constructed, as described above, and an "approximation" of the original image can be recovered, as demonstrated in Figure 3.

Figure 3: Recovered Lena image.

Consider the two images as two individual color-spaces, each occupying some abstract 3-space but displaced from each other by some unknown rotation. Each pixel in each image has its own vector representation of its color and summing all the pixels in each image yields an average vector for the color-space. Hypercomplex, or vector, cross-covariance effectively computes the average rotation quaternion over every corresponding pixel between the two images. Perfect restoration of the image can therefore only be achieved under specific conditions (for example, when the axis of rotation is perpendicular to the entire contents of both images, as demonstrated in the previous section. The type of color-space distortion used in these experiments is extreme and yet a rather good approximation of the original image is achieved. Our latest experiments, extending this and our previous work on vector phase correlation, have recovered a spatially shifted, color rotated and noise corrupted image using a reference image as a guide.

Conclusions and Future Work

Hypercomplex, vector, correlation of color images yields more than a measure of similarity. The phase and axis information acquired from the cross-covariance of two quaternion representations of color images encodes the color-space, or vector, mapping between the images. While this information can only encode an approximation of the color-space mapping in natural images, the quality of restoration of a distorted image is directly dependent on the original level of distortion.

This paper presents only an introduction to the hypercomplex correlation of vector images or signals and it is expected that more applications, as well as improvements, have yet to be discovered. Our future work is likely to concentrate on investigating the effects of large differences in luminance levels on hypercomplex cross-covariance and applying these techniques to the correlation of color and non-color images.

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Biography

Eddie Moxey is a Research Officer in the Multimedia Architectures and Applications group in the Department of Electronics Systems Engineering at the University of Essex. His research interests are hypercomplex processing of colour images and large-scale, 3D scene reconstruction from extended image sequences. After several years in industry he returned to education, graduating in 1997 with a first class B.Eng honors degree and has recently submitted a thesis for a PhD by research.

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